4.2 Vocabulary: Sets, functions, number systems

bijective function	equivalence class	poset
complement	equivalence relation	positive integers $\mathbb{Z}_{>0}$
complex numbers $\mathbb{C}$	Euclidean metric	rational numbers $\mathbb{Q}$
continuous at a point (metric spaces)	Euclidean space $\mathbb{R}^n$	real numbers $\mathbb{R}$
continuous function (metric spaces)	field	relation
continuous function (topological spaces)	fixed point	subset
convergent sequence	inf	sup
convergent series	injective function	surjective function
diameter	integers $\mathbb{Z}$	standard metric
distance between sets	interval	subsequence
distance point to set	inverse function	tolerance set
$\mathbb{E}$, the tolerance set	length norm	triangle ineq. (metric)
emptyset $\emptyset$	limit of a sequence	triangle inequality (norm)
	natural numbers	unit circle
	ordered field	unit sphere

bijective function
A *bijective function* is a function $f: X \to Y$ such that f is injective and surjective.

Cauchy-Schwarz inequality
Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a positive definite Hermitian form $\langle,\rangle: V \times V \to \mathbb{R}_{\geq 0}$ and let $\|\ \|: V \to \mathbb{R}_{\geq 0}$ be defined by $\|v\|^2 = \langle v, v\rangle$. The *Cauchy-Schwarz inequality* is

$$\text{If } x, y \in V \qquad \text{then} \quad |\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$$

complement
Let X be a set and let $A \subseteq X$. The *complement of A in X* is the set

$$A^c = \{x \in X \mid x \notin A\}.$$

complex numbers
The *complex numbers* is the $\mathbb{R}$-algebra $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ with $i^2 = -1$ and with *complex conjugation*

$$\begin{matrix} \mathbb{C} & \to & \mathbb{C} \\ c & \mapsto & \overline{c} \end{matrix} \qquad \text{given by} \qquad \overline{a+bi} = a - bi,$$

and *absolute value*

$$\begin{matrix} \mathbb{C} & \to & \mathbb{R}_{\geq 0} \\ c & \mapsto & |c| \end{matrix} \qquad \text{given by} \qquad |c|^2 = c\,\overline{c}.$$

continuous at a point (metric spaces)
Let (X, d_X) and (Y, d_Y) be metric spaces and let $a \in X$. A function $f\colon X \to Y$ is *continuous at* a if f satisfies the condition

$$\lim_{x \to a} f(x) = f(a).$$

continuous function (metric spaces)
Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is *continuous* if f satisfies

$$\text{if } a \in X \text{ then} \qquad \lim_{x \to a} f(x) = f(a).$$

emptyset
The *emptyset* $\emptyset$ is the set with no elements.

equivalence class
Let $\sim$ be an equivalence relation on a set S and let $s \in S$. The *equivalence class of* s is the set

$$[s] = \{t \in S \mid t \sim s\}.$$

equivalence relation
An *equivalence relation* on S is a relation $\sim$ on S such that

(a) if $s \in S$ then $s \sim s$,
(b) if $s_1, s_2 \in S$ and $s_1 \sim s_2$ then $s_2 \sim s_1$,
(c) if $s_1, s_2, s_3 \in S$ and $s_1 \sim s_2$ and $s_2 \sim s_3$ then $s_1 \sim s_3$.

Euclidean metric
The *Euclidean metric* is the metric on $\mathbb{R}^n$, $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, given by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}.$$

standard inner product on $\mathbb{R}^n$
The *standard inner product on* $\mathbb{R}^n$ is the functiion

$$\begin{array}{rccc} \langle , \rangle\colon & \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R} \\ & (|x_1, \ldots, x_n\rangle, |y_1, \ldots, y_n\rangle) & \mapsto & \langle x_1, \ldots, x_n \mid y_1, \ldots, y_n\rangle \end{array}$$

given by

$$\langle x_1, \ldots, x_n \mid y_1, \ldots, y_n\rangle = x_1 y_1 + \cdots + x_n y_n.$$

standard inner product on $\mathbb{C}^n$
The *standard inner product on* $\mathbb{C}^n$ is the functiion

$$\begin{array}{cccc} \langle,\rangle\colon & \mathbb{C}^n\times\mathbb{C}^n & \to & \mathbb{C} \\ & (|x_1,\ldots,x_n\rangle, |y_1,\ldots,y_n\rangle) & \mapsto & \langle x_1,\ldots,x_n \mid y_1,\ldots,y_n\rangle \end{array}$$

given by

$$\langle x_1,\ldots,x_n \mid y_1,\ldots,y_n\rangle = x_1\overline{y_1}+\cdots+x_n\overline{y_n}.$$

field
A *field* is a set $\mathbb{F}$ with functions

$$\begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & a+b \end{array} \quad\text{and}\quad \begin{array}{ccc} \mathbb{F}\times\mathbb{F} & \longrightarrow & \mathbb{F} \\ (a,b) & \longmapsto & ab \end{array}$$

such that

(Fa) If $a,b,c\in\mathbb{F}$ then $(a+b)+c=a+(b+c)$,
(Fb) If $a,b\in\mathbb{F}$ then $a+b=b+a$,
(Fc) There exists $0\in\mathbb{F}$ such that

$$\text{if } a\in\mathbb{F} \quad\text{then}\quad 0+a=a \text{ and } a+0=a,$$

(Fd) If $a\in\mathbb{F}$ then there exists $-a\in\mathbb{F}$ such that $a+(-a)=0$ and $(-a)+a=0$,
(Fe) If $a,b,c\in\mathbb{F}$ then $(ab)c=a(bc)$,
(Ff) If $a,b,c\in\mathbb{F}$ then

$$(a+b)c=ac+bc \qquad\text{and}\qquad c(a+b)=ca+cb,$$

(Fg) There exists $1\in\mathbb{F}$ such that

$$\text{if } a\in\mathbb{F} \quad\text{then}\quad 1\cdot a=a \text{ and } a\cdot 1=a,$$

(Fh) If $a\in\mathbb{F}$ and $a\neq 0$ then there exists $a^{-1}\in\mathbb{F}$ such that $aa^{-1}=1$ and $a^{-1}a=1$,
(Fi) If $a,b\in\mathbb{F}$ then $ab=ba$.

fixed point
A *fixed point of a function* $f\colon X\to X$ is

$$x\in X \qquad\text{such that}\quad f(x)=x.$$

function
A *function from S to T* is a subset $\Gamma_f\subseteq S\times T$ such that

(a) If $s\in S$ then there exists $t\in T$ such that $(s,t)\in\Gamma_f$,
(b) if $(s,t_1),(s,t_2)\in\Gamma_f$ then $t_1=t_2$.

For $s \in S$, define the notation $f(s)$ by

$$\Gamma_f = \{(s, f(s)) \mid s \in S\} \qquad \text{and write} \qquad \begin{array}{cccc} f\colon & S & \to & T \\ & s & \mapsto & f(s) \end{array}$$

Abusing semantics, the set Γ_f is *the graph of* f.

image (function)
Let $f\colon S \to T$ be a function. The *image* of f is

$$\mathrm{im}(f) = \{f(s) \mid s \in S\}.$$

inf, or **infimum**, or **greatest lower bound**
Let S be a poset and let E be a subset of S. A *infimum of E in S*, or *greatest lower bound of E in S*, is an element $\inf(E) \in S$ such that

(a) $\inf(E)$ is a lower bound of E in S, and
(b) If $l \in S$ is a lower bound of E in S then $l \le \inf(E)$.

injective function
Let X and Y be sets. A *injective function* from X to Y is a function $f\colon X \to Y$ such that

$$\text{if } x_1, x_2 \in X \text{ and } f(x_1) = f(x_2) \quad \text{then} \quad x_1 = x_2.$$

integers $\mathbb{Z}$
The *integers* is the set

$$\mathbb{Z} = (-\mathbb{Z}_{>0}) \cup \{0\} \cup \mathbb{Z}_{>0} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

with addition given by the addition in $\mathbb{Z}_{>0}$ and the conditions

(a) If $z \in \mathbb{Z}$ then $0 + z = z + 0 = z$,
(b) If $x, y, z \in \mathbb{Z}$ then $(x + y) + z = x + (y + z)$,
(c) If $x \in \mathbb{Z}_{>0}$ then $x + (-x) = (-x) + x = 0$.

The ordering on $\mathbb{Z}$ is given by

$$x < y \qquad \text{if there exists } p \in \mathbb{Z} > 0 \text{ such that } x + p = y.$$

interval

inverse function
Let X and Y be sets and let $f\colon X \to Y$ be a function from X to Y. The *inverse function to f* is the function $f^{-1}\colon Y \to X$ such that

$$f^{-1} \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}_Y.$$

matrix
Let $t, s \in \mathbb{Z}_{>0}$ and let $\mathbb{F}$ be a field. A *t by s matrix with entries in* $\mathbb{F}$ is a function

$$A\colon \{1, \ldots, t\} \times \{1, \ldots, s\} \to \mathbb{F}.$$

metric
Let X be a set. A *metric on X* is a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$ such that

(a) If $x \in X$ then $d(x, x) = 0$,
(b) If $x, y \in X$ and $d(x, y) = 0$ then $x = y$,
(c) If $x, y \in X$ then $d(x, y) = d(y, x)$,
(d) If $x, y, z \in X$ then $d(x, y) \leq d(x, z) + d(z, y)$.

ordered field
An *ordered field* is a field $\mathbb{F}$ with a total order $\leq$ such that

(OFa) If $a, b, c \in \mathbb{F}$ and $a \leq b$ then $a + c \leq b + c$,
(OFb) If $a, b \in \mathbb{F}$ and $a \geq 0$ and $b \geq 0$ then $ab \geq 0$.

partition of a set
A *partition of a set* S is a collection $\mathcal{P}$ of subsets of S such that

(a) If $s \in S$ then there exists $P \in \mathcal{P}$ such that $s \in P$, and
(b) If $P_1, P_2 \in \mathcal{P}$ and $P_1 \cap P_2 \neq \emptyset$ then $P_1 = P_2$.

poset
A *poset*, or *partially ordered set*, is a set S with a relation $\leq$ on S such that

(a) If $x \in A$ then $x \leq x$,
(b) If $x, y, z \in S$ and $x \leq y$ and $y \leq z$ then $x \leq z$, and
(c) If $x, y \in S$ and $x \leq y$ and $y \leq x$ then $x = y$.

positive integers $\mathbb{Z}_{>0}$
The *positive integers* is the set

$$\mathbb{Z}_{>0} = \{1, 2, 3, \ldots\}, \qquad \text{where} \quad \begin{matrix} 2 = 1 + 1, \\ 3 = 2 + 1, \\ 4 = 3 + 1, \\ 5 = 4 + 1, \\ \vdots \end{matrix}$$

and with addition given by concatention so that, for example, $2+3 = 1+1 \quad + \quad 1+1+1 = 5$. The ordering on the positive integers is given by

$$x < y \quad \text{if there exists } p \in \mathbb{Z}_{>0} \text{ such that } x+p = y.$$

rational numbers $\mathbb{Q}$
The *rational numbers* is the set

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z} \text{ with } b \neq 0 \right\} \qquad \text{with} \quad \frac{a}{b} = \frac{c}{d} \text{ if } ad = bc,$$

with the functions

$$\begin{matrix} \mathbb{Q} \times \mathbb{Q} & \to & \mathbb{Q} \\ \left(\frac{a}{b}, \frac{c}{d}\right) & \mapsto & \frac{a}{b} + \frac{c}{d} \end{matrix} \qquad \text{and} \qquad \begin{matrix} \mathbb{Q} \times \mathbb{Q} & \to & \mathbb{Q} \\ \left(\frac{a}{b}, \frac{c}{d}\right) & \mapsto & \frac{a}{b} \cdot \frac{c}{d} \end{matrix}$$

given by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \qquad \text{and} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd},$$

real numbers $\mathbb{R}$

relation
A *relation* $\sim$ *on* S is a subset $R_\sim$ of $S \times S$. Write $s_1 \sim s_2$ if the pair (s_1, s_2) is in the subset $R_\sim$ so that

$$R_\sim = \{(s_1, s_2) \in S \times S \mid s_1 \sim s_2\}.$$

subset
Let X be a set. A *subset* of X is a set A such that

$$\text{if } a \in A \quad \text{then} \quad a \in X.$$

Write $A \subseteq X$ if A is a subset of X.

sup
Let S be a poset and let E be a subset of S. A *supremum*, or*least upper bound of E in S* is an element $\sup(E) \in S$ such that

(a) $\sup(E)$ is a upper bound of E in S, and
(b) If $b \in S$ is a upper bound of E in S then $\sup(E) \leq b$.

surjective function
Let X and Y be sets. A *surjective function* from X to Y is a function $f\colon X \to Y$ such that

$$\text{if } y \in Y \quad \text{then} \quad \text{there exists } x \in X \text{ such that } f(x) = y.$$

standard inner product
The standard inner product on $\mathbb{C}^n$ is $\langle , \rangle \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = x_1\overline{y_1} + \cdots x_n\overline{y_n}.$$

standard metric on $\mathbb{C}$
The *standard metric on* $\mathbb{C}$ is the metric

$$d\colon \mathbb{C} \times \mathbb{C} \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(z, w) = |z - w|.$$

triangle inequality for a function $d\colon X \times X \to \mathbb{R}_{\geq 0}$
Let X be a set and let $d\colon X \times X \to \mathbb{R}_{\geq 0}$ be a function. The function d satisfies the *triangle inequality* if d satisfies

$$\text{if } x, y, z \in X \quad \text{then} \quad d(x, y) \leq d(x, z) + d(z, y).$$

triangle inequality for $\|\ \|\colon X \to \mathbb{R}_{\geq 0}$
Let X be a vector space over a field $\mathbb{K}$ and let $\|\ \|\colon X \to \mathbb{R}_{\geq 0}$ be a function. The function $\|\ \|\colon X \to \mathbb{R}_{\geq 0}$ satisfies the *triangle inequality* if $\|\ \|\colon X \to \mathbb{R}_{\geq 0}$ satisfies

$$\text{if } x, y \in X \quad \text{then} \quad \|x + y\| \leq \|x\| + \|y\|.$$

unit circle
The *unit circle* is the set

$$\begin{aligned} S^1 &= \{z \in \mathbb{C} \mid |z| = 1\} \\ &= \{e^{2\pi i\theta} \mid \theta \in \mathbb{R}, 0 \leq \theta < 2\pi\} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } x^2 + y^2 = 1\}. \end{aligned}$$

unit sphere
Let $(V, \|\ \|)$ be a normed vector space. The *unit sphere in V* is

$$S = \{v \in V \mid \|v\| = 1\}.$$

References

[Mor] A.O. Morris, *Linear algebra, An introduction*, Chapman and Hall, 1982 ISBN:0-412-38100-1