4 Vocabulary

4.1 Vocabulary: Linear algebra

adjoint
adjoint with respect to $\langle , \rangle$
basis (vector space)
Cauchy-Schwarz ineq.
change of basis matrix
characteristic polynomial (matrix)
characteristic polynomial (linear transformation)
determinant
diagonal matrix
diagonalizable matrix
direct sum (vector spaces)
dual space (vector space)
dual space (normed vector space)????
eigenspace
eigenspectrum????
Euclidean space $\mathbb{R}^n$???
eigenspace (matrix)
eigenspace (linear transformation)
eigenvalue (matrix)
eigenvalue (linear transformation)
eigenvector (matrix)
eigenvector (linear transformation)
elementary matrix
Fourier coefficients
Fourier series
$GL_n(\mathbb{F})$
Gram-Schmidt process
Hilbert space
inner product
image (function)
image (matrix)
image (linear transformation)
inverse (matrix)
inverse (linear transformation)
invertible matrix
kernel (matrix)
kernel (linear transformation)
length norm
linearly independent
linear transformation
$M_{t\times s}(\mathbb{F})$
matrix
matrix of a linear transformation with respect to bases
metric
metric space???
norm $\| \ \|$
normed vector space
norm metric
operator norm
orthogonal complement
orthonormal sequence
orthonormal basis
product (for matrices)
product (for linear transformations)
root matrix
self adjoint operator
singlun matrix
span
standard metric
subspace
triangle ineq. (metric)
triangle inequality (norm)
unitary operator
vector space

adjoint
Let $T\colon V \to W$ be an $\mathbb{F}$-linear transformation. Let W^* be the dual space of W and let V^* be

the dual space of V. The *adjoint of* T is the function

$$T^*\colon W^* \to V^* \qquad \text{given by} \quad (T^*\psi)(v) = (\psi \circ T)(v).$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "T^*\psi"'] & W \arrow[d, "\psi"] \\ & \mathbb{K} \end{tikzcd}$$

adjoint with respect to $\langle,\rangle$

Let V be an $\mathbb{F}$-vector space with a nondegenerate sesquilinear form $\langle,\rangle\colon V\times V \to \mathbb{F}$. Let $f\colon V \to V$ be a linear transformation. The *adjoint of f with respect to $\langle,\rangle$* is the linear transformation $f^*\colon V \to V$ determined by

$$\text{if } x,y \in V \text{ then} \quad \langle f(x), y\rangle = \langle x, f^*(y)\rangle.$$

basis (vector space)

Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. A *basis* of V is a subset $B \subseteq V$ such that

(a) $\mathbb{F}\text{-span}(B) = V$,
(b) B is linearly independent,

where

$$\mathbb{F}\text{-span}(B) = \{a_1b_1 + \cdots + a_\ell b_\ell \mid \ell \in \mathbb{Z}_{>0},\ b_1,\ldots,b_\ell \in B,\ a_1,\ldots,a_\ell \in \mathbb{F}\},$$

and B is *linearly independent* if B satisfies

$$\begin{gathered}\text{if } \ell \in \mathbb{Z}_{>0} \text{ and } b_1,\ldots,b_\ell \in B \text{ and } a_1,\ldots,a_\ell \in \mathbb{F}, \text{ and} \\ a_1b_1 + \cdots + a_\ell b_\ell = 0 \quad \text{then} \quad a_1 = 0,\ a_2 = 0,\ \ldots,\ a_\ell = 0.\end{gathered}$$

Cauchy-Schwarz inequality

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ with a positive definite Hermitian form $\langle,\rangle\colon V\times V \to \mathbb{R}_{\ge 0}$ and let $\|\ \|\colon V \to \mathbb{R}_{\ge 0}$ be defined by $\|v\|^2 = \langle v,v\rangle$. The *Cauchy-Schwarz inequality* is

$$\text{If } x,y \in V \qquad \text{then} \quad |\langle x,y\rangle| \le \|x\|\cdot\|y\|.$$

change of basis matrix

Let $n \in \mathbb{Z}_{>0}$. Let V be an $\mathbb{F}$-vector space and assume that $B = \{b_1,\ldots,b_n\}$ is a basis of V and $C = \{c_1,c_2,\ldots,c_n\}$ be a basis of V. The change of basis matrix from B to C is the matrix

$$[I]_{CB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \qquad \text{where} \qquad \begin{aligned} c_1 &= a_{11}b_1 + a_{21}b_2 + \cdots + a_{n1}b_n, \\ c_2 &= a_{12}b_1 + a_{22}b_2 + \cdots + a_{n2}b_n, \\ &\ \vdots \\ c_n &= a_{1n}b_1 + a_{2n}b_2 + \cdots + a_{nn}b_n. \end{aligned}$$

characteristic polynomial (matrix)
Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$. Let $A \in M_{n\times n}(\mathbb{F})$ and let $\lambda \in \mathbb{F}$. The *characteristic polynomial of A* is

$$\det(t-A) \qquad \text{(an element of } \mathbb{F}[t]).$$

characteristic polynomial (linear transformation)
Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space, let $T\colon V\to V$ be a linear transformation and let $\lambda \in \mathbb{F}$. The *characteristic polynomial of A* is

$$\det(t-T) \qquad \text{(an element of } \mathbb{F}[t]).$$

determinant
Let $n \in \mathbb{Z}_{>0}$ and let $\mathbb{F}$ be a field. For $i,j\in\{1,\ldots,n\}$ let $E_{ij}\in M_{n\times n}(\mathbb{F})$ be the matrix with 1 in the (i,j) entry and all other entries 0. For $c,d_1,\ldots,d_n\in\mathbb{F}$ and $i,j\in\{1,\ldots,n\}$ with $i\neq j$ let

$$x_{ij}(c) = 1 + cE_{ij} \qquad \text{and} \qquad h(d_1,\ldots,d_n) = d_1E_{11}+\cdots+d_nE_{nn}.$$

The *determinant* is the function $\det\colon M_{n\times n}(\mathbb{F})\to\mathbb{F}$ determined by the conditions

(a) If $A,B\in M_{n\times n}(\mathbb{F})$ then $\det(AB)=\det(A)\det(B)$,
(b) If $c\in\mathbb{F}$ and $i,j\in\{1,\ldots,n\}$ and $i\neq j$ then $\det(x_{ij}(c))=1$,
(c) If $d_1,\ldots,d_n\in\mathbb{F}$ then $\det(h(d_1,\ldots,d_n))=d_1d_2\cdots d_n$.

diagonal matrix
Let $n \in \mathbb{Z}_{>0}$ and let $\mathbb{F}$ be a field. For $i,j\in\{1,\ldots,n\}$ let $E_{ij}\in M_{n\times n}(\mathbb{F})$ be the matrix with 1 in the (i,j) entry and all other entries 0. A *diagonal matrix* is a matrix

$$h(d_1,\ldots,d_n) = d_1E_{11}+\cdots+d_nE_{nn}, \qquad \text{with } d_1,\ldots,d_n\in\mathbb{F}.$$

diagonalizable matrix
Let $n \in \mathbb{Z}_{>0}$ and let $\mathbb{F}$ be a field. For $i,j\in\{1,\ldots,n\}$ let $E_{ij}\in M_{n\times n}(\mathbb{F})$ be the matrix with 1 in the (i,j) entry and all other entries 0. Let $A\in M_{n\times n}(\mathbb{F})$. The matrix A is $\mathbb{F}$-*diagonalizable*, or *semisimple*, if there exist $P\in GL_n(\mathbb{F})$ and $d_1,\ldots,d_n\in\mathbb{F}$ such that

$$P^{-1}AP = h(d_1,\ldots,d_n), \qquad \text{where} \quad h(d_1,\ldots,d_n)=d_1E_{11}+\cdots+d_nE_{nn}.$$

direct sum (vector spaces)
Let V be an $\mathbb{F}$-vector space with basis $B=\{b_1,\ldots,b_k\}$ and let W be an $\mathbb{F}$-vector space with basis $C=\{c_1,\ldots,c_\ell\}$. The *direct sum of V and W* is the $\mathbb{F}$-vector space

$$V\oplus W \qquad \text{with basis} \qquad B\cup C=\{b_1,\ldots,b_k,c_1,\ldots,c_\ell\}.$$

dual space (vector space)
Let $\mathbb{F}$ be a field and let W be an $\mathbb{F}$-vector space. The *dual space* to W is the vector space

$$W^* = \mathrm{Hom}(W, \mathbb{F}) = \{\varphi\colon W \to \mathbb{F} \mid \varphi \text{ is a linear transformation}\}$$

with addition and scalar multiplication given by

$$(\varphi_1 + \varphi_2)(w) = \varphi_1(w) + \varphi_2(w) \qquad \text{and} \qquad (c\varphi)(w) = c \cdot \varphi(w).$$

for $\varphi, \varphi_1, \varphi_2 \in W^*$, $w \in W$ and $c \in \mathbb{F}$.

eigenspace (matrix)
Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$. Let $A \in M_{n\times n}(\mathbb{F})$ and let $\lambda \in \mathbb{F}$. The λ-*eigenspace of* A is

$$V_\lambda = \ker(A - \lambda) = \{v \in \mathbb{F}^n \mid Av = \lambda v\}.$$

eigenspace (linear transformation)
Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space, let $T\colon V \to V$ be a linear transformation and let $\lambda \in \mathbb{F}$. The λ-*eigenspace of* $T\colon V \to V$ is

$$V_\lambda = \ker(T - \lambda) = \{v \in V \mid Tv = \lambda v\}.$$

eigenvalue (matrix)
Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$. Let $A \in M_{n\times n}(\mathbb{F})$. An *eigenvalue of* A is $\lambda \in \mathbb{F}$ such that

$$\ker(A - \lambda) \neq 0.$$

eigenvalue (linear transformation)
Let $T\colon V \to V$ be an $\mathbb{F}$-linear transformation and let $\lambda \in \mathbb{F}$. An *eigenvalue of* T is $\lambda \in \mathbb{F}$ such that

$$\ker(T - \lambda) \neq 0.$$

eigenvector (matrix)
Let $n \in \mathbb{Z}_{>0}$. Let $A \in M_{n\times n}(\mathbb{F})$ and let $\lambda \in \mathbb{F}$. An $\mathbb{F}$-*eigenvector of* A of eigenvalue λ is a

$$\text{nonzero} \quad v \in \ker(A - \lambda).$$

eigenvector (linear transformation)
Let $T\colon V \to V$ be an $\mathbb{F}$-linear transformation and let $\lambda \in \mathbb{F}$. An $\mathbb{F}$-*eigenvector of* T of eigenvalue λ is a

$$\text{nonzero} \quad v \in \ker(T - \lambda).$$

elementary matrix
Let $n \in \mathbb{Z}_{>0}$. Let $\mathbb{F}$ be a field and let $\mathbb{F}^\times = \mathbb{F} - \{0\}$. For $i, j \in \{1, \ldots, n\}$ let $E_{ij} \in M_{n\times n}(\mathbb{F})$ have (i, j) entry 1 and all other entries 0. An *elementary matrix* is one of the matrices

$$x_{ij}(c) = 1 + cE_{ij}, \qquad \text{with } c \in \mathbb{F} \text{ and } i \neq j,$$

or

$$s_{ij} = 1 + E_{ij} + E_{ji} - E_{ii} - E_{jj}, \qquad \text{with } i \neq j,$$

or

$$h(d_1, \ldots, d_n) = d_1E_{11} + \cdots + d_nE_{nn}, \qquad \text{with } d_1, \ldots, d_n \in \mathbb{F}^\times.$$

Euclidean metric
The *Euclidean metric* is the metric on $\mathbb{R}^n$, $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, given by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots (x_n - y_n)^2}.$$

$GL_n(\mathbb{F})$
Let $\mathbb{F}$ be a field and let $n \in \mathbb{Z}_{>0}$. The *general linear group* is

$$GL_n(\mathbb{F}) = \{P \in M_{n\times n}(\mathbb{F}) \mid P \text{ is invertible}\}$$

with the function $GL_n(\mathbb{F}) \times GL_n(\mathbb{F}) \to GL_n(\mathbb{F})$ given by matrix multiplication.

Gram-Schmidt process
Let $(V, \langle , \rangle)$ be a positive definite Hermitian inner product space. Let $v_1, v_2, \ldots$ be a sequence of linearly independent vectors in V. The *Gram-Schmidt process* is the use of the vectors $v_1, v_2, \ldots$ to construct the vectors $a_1, a_2, \ldots$ in V given by

$$a_1 = \frac{v_1}{\|v_1\|} \qquad \text{and} \qquad a_{n+1} = \frac{v_{n+1} - \langle v_{n+1}, a_1\rangle a_1 - \cdots - \langle v_{n+1}, a_n\rangle a_n}{\|v_{n+1} - \langle v_{n+1}, a_1\rangle a_1 - \cdots - \langle v_{n+1}, a_n\rangle a_n\|}.$$

Hermitian (matrix)
Let $A \in M_n(\mathbb{C})$. The matrix A is *Hermitian* if

$$A = \overline{A}^t.$$

Hilbert space
A *Hilbert space* is a positive definite Hermitian inner product space $(V, \langle, \rangle)$ which is a complete metric space with the metric $d\colon V \times V \to \mathbb{R}_{\geq 0}$ given by

$$d(x, y) = \|x - y\|, \qquad \text{where} \qquad \|x - y\| = \sqrt{\langle x - y, x - y\rangle}.$$

image (function)
Let $f\colon S \to T$ be a function. The *image* of f is

$$\mathrm{im}(f) = \{f(s) \mid s \in S\}.$$

image (matrix)
Let $\mathbb{F}$ be a field. Let $t, s \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{F})$. The *image* of A is

$$\mathrm{im}(A) = \{Av \mid v \in \mathbb{F}^s\}.$$

image (linear transformation)
Let $T\colon V \to W$ be an $\mathbb{F}$-linear transformation. The *image* of T is

$$\mathrm{im}(T) = \{T(v) \mid v \in V\}.$$

inner product
Whenever anyone uses this word you should respond, "Do you mean Hermitian or symmetric, or positive definite, or nonisotropic, nondegenerate, or sesquilinear, or just bilinear?"

$L^2(X)$
Let (X, μ) be a measure space. For $f\colon X \to \mathbb{C}$ let $\|f\|_2 = \sqrt{\int_X |f(x)|^2 dx}$ if it exists in $\mathbb{R}_{\geq 0}$.

$$L^2(X) = \{f\colon X \to \mathbb{C} \mid \|f\|_2 \text{ exists in } \mathbb{R}_{\geq 0}\},$$

with inner product $\langle,\rangle\colon L^2(X) \times L^2(X) \to \mathbb{C}$ given by

$$\langle f, g\rangle = \int_X f(x)\overline{g(x)} d\mu.$$

inverse (matrix)
Let $n \in \mathbb{Z}_{>0}$ and let $A \in M_n(\mathbb{F})$. The *inverse of* A is a matrix $A^{-1} \in M_n(\mathbb{F})$ such that

$$AA^{-1} = 1 \qquad \text{and} \qquad A^{-1}A = 1.$$

inverse (linear transformation)
Let $T\colon V \to V$ be a linear transformation. The *inverse of* T is a linear transformation $T^{-1}\colon V \to V$ such that

$$TT^{-1} = 1 \qquad \text{and} \qquad T^{-1}T = 1.$$

invertible matrix
Let $n \in \mathbb{Z}_{>0}$ and let $A \in M_n(\mathbb{F})$. The matrix A is *invertible* if there exists $A^{-1} \in M_n(\mathbb{F})$ such that

$$AA^{-1} = 1 \qquad \text{and} \qquad A^{-1}A = 1.$$

kernel (matrix)
Let $\mathbb{F}$ be a field and let $s, t \in \mathbb{Z}_{>0}$. Let $A \in M_{t\times s}(\mathbb{F})$. The *kernel* of A is

$$\ker(A) = \{v \in \mathbb{F}^s \mid Av = 0\}.$$

kernel (linear transformation)
Let $T\colon V \to W$ be a linear transformation. The *kernel* of T is

$$\ker(T) = \{v \in V \mid T(v) = 0\}.$$

length norm
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let V be a vector space over $\mathbb{K}$ with a positive definite Hermitian form $\langle,\rangle\colon V \times V \to \mathbb{K}$. The *length norm* on V is the function

$$\begin{matrix} V & \to & \mathbb{R}_{\ge 0} \\ v & \mapsto & \|v\| \end{matrix} \qquad \text{determined by} \qquad \|v\|^2 = \langle v, v\rangle.$$

linearly independent
Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Let B be a subset of V. The set B is *linearly independent*, or *has no relations*, if B satisfies

$$\begin{gathered}\text{if } \ell \in \mathbb{Z}_{>0} \text{ and } b_1, \ldots, b_\ell \in B \text{ and } a_1, \ldots, a_\ell \in \mathbb{F}, \text{ and} \\ a_1 b_1 + \cdots + a_\ell b_\ell = 0 \quad \text{then} \quad a_1 = 0,\ a_2 = 0,\ \ldots,\ a_\ell = 0.\end{gathered}$$

linear transformation
Let V and W be $\mathbb{F}$-vector spaces. A *linear transformation from V to W* is a function $T\colon V \to W$ such that

(a) if $v_1, v_2 \in V$ then $T(v_1 + v_2) = T(v_1) + T(v_2)$,
(b) If $v \in V$ and $c \in \mathbb{F}$ then $T(cv) = cT(v)$.

$M_{t\times s}(\mathbb{F})$
Let $t, s \in \mathbb{Z}_{>0}$ and let $\mathbb{F}$ be a field. The *vector space of $t \times s$ matrices with entries in $\mathbb{F}$* is the set $M_{t\times s}(\mathbb{F})$ of $t \times s$ matrices with entries in $\mathbb{F}$ with the functions

$$\begin{matrix} M_{t\times s}(\mathbb{F}) \times M_{t\times s}(\mathbb{F}) & \to & M_{t\times s}(\mathbb{F}) \\ (A, B) & \mapsto & A + B \end{matrix} \qquad \text{and} \qquad \begin{matrix} \mathbb{F} \times M_{t\times s}(\mathbb{F}) & \to & M_{t\times s}(\mathbb{F}) \\ (c, A) & \mapsto & cA \end{matrix}$$

given by

$$(A + B)(i, j) = A(i, j) + B(i, j) \qquad \text{and} \qquad (cA)(i, j) = c \cdot A(i, j),$$

for $i \in \{1, \ldots t\}$ and $j \in \{1, \ldots, s\}$.

matrix
Let $t, s \in \mathbb{Z}_{>0}$ and let $\mathbb{F}$ be a field. A *t by s matrix with entries in $\mathbb{F}$* is a function

$$A\colon \{1,\ldots,t\}\times\{1,\ldots,s\}\to\mathbb{F}.$$

matrix of a linear transformation with respect to bases
Let $s, t \in \mathbb{Z}_{>0}$. Let $T\colon V\to W$ be an $\mathbb{F}$-linear transformation. Assume that $B = \{b_1,\ldots,b_s\}$ is a basis of V and $C = \{c_1, c_2, \ldots, c_t\}$ be a basis of W. The *matrix of T with respect to the bases B and C* is

$$[T]_{CB} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ a_{21} & a_{22} & \cdots & a_{2s} \\ \vdots & & & \vdots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix} \qquad \text{where} \qquad \begin{array}{l} T(b_1) = a_{11}c_1 + a_{21}c_2 + \cdots + a_{t1}c_t, \\ T(b_2) = a_{12}c_1 + a_{22}c_2 + \cdots + a_{t2}c_t, \\ \quad\vdots \\ T(b_s) = a_{1s}c_1 + a_{2s}c_2 + \cdots + a_{ts}c_t. \end{array}$$

metric
Let X be a set. A *metric on X* is a function $d\colon X\times X\to\mathbb{R}_{\geq 0}$ such that

(a) If $x \in X$ then $d(x,x) = 0$,
(b) If $x, y \in X$ and $d(x,y) = 0$ then $x = y$,
(c) If $x, y \in X$ then $d(x,y) = d(y,x)$,
(d) If $x, y, z \in X$ then $d(x,y) \leq d(x,z) + d(z,y)$.

norm
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let V be a $\mathbb{K}$-vector space. A *norm* on V is a function $\|\ \|\colon V\to\mathbb{R}_{\geq 0}$ such that

(a) If $x, y \in V$ then $\|x+y\| \leq \|x\| + \|y\|$,
(b) If $c \in \mathbb{K}$ and $v \in V$ then $\|cv\| = |c|\,\|v\|$,
(c) If $v \in V$ and $\|v\| = 0$ then $v = 0$.

normed vector space
Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. A *normed vector space* is a $\mathbb{K}$-vector space V with a function $\|\ \|\colon V\to\mathbb{R}_{\geq 0}$ such that

(a) If $x, y \in V$ then $\|x+y\| \leq \|x\| + \|y\|$,
(b) If $c \in \mathbb{K}$ and $v \in V$ then $\|cv\| = |c|\,\|v\|$,
(c) If $v \in V$ and $\|v\| = 0$ then $v = 0$.

norm metric
Let $(V, \|\ \|)$ be a normed vector space. The *norm metric* on V is the function

$$d\colon V\times V\to\mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(x,y) = \|x-y\|.$$

operator norm
Let $(V, \| \|_V)$ and $(W, \| \|_W$ be normed vector spaces and let $T\colon V \to W$ be a linear transformation. The *operator norm* of T is

$$\|T\| = \sup \left\{ \frac{\|Tx\|_W}{\|x\|_V} \mid x \in V \right\}.$$

orthogonal (matrix)
Let $A \in M_n(\mathbb{C})$. The matrix A is *orthogonal* if

$$AA^t = 1.$$

orthogonal complement
Let $(V, \langle\,,\,\rangle)$ be an inner product space and let $W \subseteq V$ be a subspace of V. The *orthogonal complement of W in V* is

$$W^\perp = \{v \in V \mid \text{if } w \in W \text{ then } \langle v, w\rangle = 0\}.$$

orthonormal sequence
Let $\mathbb{F}$ be a field and let $(V, \langle\,,\,\rangle)$ be an $\mathbb{F}$-vector space with a sequilinear form. An *orthonormal sequence* in V is a sequence $(b_1, b_2, \ldots)$ in V such that

$$\text{if } i, j \in \mathbb{Z}_{>0} \qquad \text{then} \qquad \langle b_i, b_j\rangle = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

orthonormal basis
Let H be a separable Hilbert space. An *orthonormal basis of H* is a subset $A \subseteq H$ such that A is countable, A is orthonormal, and $\overline{\mathrm{span}(A)} = H$.

product (for matrices)
Let $s, n, t \in \mathbb{Z}_{>0}$ and let $A_1 \in M_{n\times s}(\mathbb{F})$ and $A_2 \in M_{t\times n}(\mathbb{F})$. The *product of A_2 with A_1* is the matrix $A_2A_1 \in M_{t\times s}(\mathbb{F})$ given by

$$(A_2A_1)(i,j) = \sum_{\ell=1}^{n} A_2(i,\ell)A_1(\ell,j) \qquad \text{for } i \in \{1,\ldots,t\} \text{ and } j \in \{1,\ldots,s\}.$$

product (for linear transformations)
Let $T_1\colon V \to W$ and $T_2\colon W \to Z$ be $\mathbb{F}$-linear transformations. The *composition of T_2 and T_1*, or *product of T_2 and T_1* is the composition

$$T_2T_1 = T_2 \circ T_1, \qquad \text{so that } T_2T_1(v) = T_2(T_1(v)) \text{ for } v \in V.$$

root matrix
Let $n \in \mathbb{Z}_{>0}$. A *root matrix* for $GL_n(\mathbb{F})$ is a matrix

$$x_{ij}(c) = 1 + cE_{ij}, \qquad \text{with } c \in \mathbb{F}$$

and $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

self adjoint operator
Let H be a Hilbert space. A *self adjoint operator on H* is

$$\text{a bounded linear operator } T \colon H \to H \quad \text{such that} \quad T = T^*,$$

where T^* is the adjoint of T.

singlun matrix
Let $t, s \in \mathbb{Z}_{>0}$. For $i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, s\}$ the *singlun matrix* $E_{ij} \in M_{t\times s}(\mathbb{F})$ is the matrix with 1 in the (i, j) entry and 0 elsewhere.

span
Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Let B be a subset of V. The *span of B*, or *subspace generated by B*, is

$$\mathbb{F}\text{-span}(B) = \{a_1b_1 + \cdots + a_\ell b_\ell \mid \ell \in \mathbb{Z}_{>0},\ b_1, \ldots, b_\ell \in B,\ a_1, \ldots, a_\ell \in \mathbb{F}\},$$

standard inner product on $\mathbb{R}^n$
The *standard inner product on $\mathbb{R}^n$* is the functiion

$$\begin{array}{rccc} \langle , \rangle \colon & \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R} \\ & (|x_1, \ldots, x_n\rangle, |y_1, \ldots, y_n\rangle) & \mapsto & \langle x_1, \ldots, x_n \mid y_1, \ldots, y_n\rangle \end{array}$$

given by

$$\langle x_1, \ldots, x_n \mid y_1, \ldots, y_n\rangle = x_1y_1 + \cdots + x_ny_n.$$

standard inner product on $\mathbb{C}^n$
The *standard inner product on $\mathbb{C}^n$* is the functiion

$$\begin{array}{rccc} \langle , \rangle \colon & \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{C} \\ & (|x_1, \ldots, x_n\rangle, |y_1, \ldots, y_n\rangle) & \mapsto & \langle x_1, \ldots, x_n \mid y_1, \ldots, y_n\rangle \end{array}$$

given by

$$\langle x_1, \ldots, x_n \mid y_1, \ldots, y_n\rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n}.$$

standard metric
The *standard metric* is the metric

$$d \colon \mathbb{C} \times \mathbb{C} \to \mathbb{R}_{\geq 0} \qquad \text{given by} \qquad d(z, w) = |z - w|.$$

subspace
Let V be an $\mathbb{F}$-vector space. A *subspace of* V is a subset W of V such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,
(b) If $w \in W$ and $c \in \mathbb{F}$ then $cw \in W$.

symmetric (matrix)
Let $A \in M_n(\mathbb{F})$. The matrix A is *symmetric* if

$$A = A^t.$$

triangle inequality for a function $d\colon X \times X \to \mathbb{R}_{\ge 0}$
Let X be a set and let $d\colon X \times X \to \mathbb{R}_{\ge 0}$ be a function. The function d satisfies the *triangle inequality* if d satisfies

$$\text{if } x, y, z \in X \quad \text{then} \quad d(x,y) \le d(x,z) + d(z,y).$$

triangle inequality for $\| \ \|\colon X \to \mathbb{R}_{\ge 0}$
Let X be a vector space over a field $\mathbb{K}$ and let $\| \ \|\colon X \to \mathbb{R}_{\ge 0}$ be a function. The function $\| \ \|\colon X \to \mathbb{R}_{\ge 0}$ satisfies the *triangle inequality* if $\| \ \|\colon X \to \mathbb{R}_{\ge 0}$ satisfies

$$\text{if } x, y \in X \quad \text{then} \quad \|x+y\| \le \|x\| + \|y\|.$$

unitary (matrix)
Let $A \in M_n(\mathbb{C})$. The matrix A is *unitary* if

$$A\overline{A}^t = 1.$$

unitary operator
Let H be a Hilbert space. A *unitary operator on* H is

$$\text{a bounded linear operator } T\colon H \to H \quad \text{such that} \quad TT^* = T^*T = I,$$

where T^* is the adjoint of T and I is the identity operator on H.

vector space
Let $\mathbb{F}$ be a field. A $\mathbb{F}$-*vector space* is a set V with functions

$$\begin{array}{ccc} V \times V & \to & V \\ (v_1, v_2) & \mapsto & v_1 + v_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbb{F} \times V & \to & V \\ (c, v) & \mapsto & cv \end{array}$$

(*addition* and *scalar multiplication*) such that

(a) If $v_1, v_2, v_3 \in V$ then $(v_1 + v_2) + v_3 = v_1 + (v_2 + v_3)$,

(b) There exists $0 \in V$ such that if $v \in V$ then $0 + v = v$ and $v + 0 = v$,

(c) If $v \in V$ then there exists $-v \in V$ such that $v + (-v) = 0$ and $(-v) + v = 0$,

(d) If $v_1, v_2 \in V$ then $v_1 + v_2 = v_2 + v_1$,

(e) If $c \in \mathbb{F}$ and $v_1, v_2 \in V$ then $c(v_1 + v_2) = cv_1 + cv_2$,

(f) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $(c_1 + c_2)v = c_1v + c_2v$,

(g) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ then $c_1(c_2v) = (c_1c_2)v$,

(h) If $v \in V$ then $1v = v$.