2.9 Week 9 Tutorial: Conic sections

The *cone* is the set of points

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}. \qquad PICTURE$$

Consider a plane: Fix $a, b, c, d \in \mathbb{R}$ such that $a^2 + b^2 + c^2 = 1$ and let

$$P = \{(x, y, z) \mid ax + by + cz = d\}. \qquad PICTURE$$

Since $a^2 + b^2 + c^2 = 1$ then (a, b, c) is a unit vector that is perpendicular to the plane P. Then a rotation in $\mathbb{R}^3$ will move the normal vector (a, b, c) to the vector $(0, 0, 1)$ in the new coordinates u, v, w where

$$\begin{array}{l} x = \frac{a^2c+b^2}{a^2+b^2}u + \frac{ab(c-1)}{a^2+b^2}v + aw \\ y = \frac{ab(c-1)}{a^2+b^2}u + \frac{b^2c+a^2}{a^2+b^2}v + bw \\ z = -au - bv - cw \end{array} \qquad \text{and} \qquad \begin{array}{l} u = \frac{a^2c+b^2}{a^2+b^2}x + \frac{ab(c-1)}{a^2+b^2}y - az \\ v = \frac{ab(c-1)}{a^2+b^2}x + \frac{b^2c+a^2}{a^2+b^2}y - bz \\ w = ax + by + cz \end{array}$$

The first formulas were obtained by asking Google "what is the rotation that takes $(a, b, \sqrt{1 - a^2 + b^2})$ to $(0, 0, 1)$". Since this is an orthogonal transformation its inverse is given by the transpose matrix. To check that this is right, see that the equation of the plane becomes

$$\begin{aligned} d &= ax + by + cz \\ &= a\Big(\frac{a^2c + b^2}{a^2 + b^2}u + \frac{ab(c-1)}{a^2 + b^2}v + aw\Big) + b\Big(\frac{ab(c-1)}{a^2 + b^2}u + \frac{b^2c + a^2}{a^2 + b^2}v + bw\Big) + c(-au - bv - cw) \\ &= a\Big(\frac{a^2c + b^2 + b^2c - b^2}{a^2 + b^2} - c\Big)u + b\Big(\frac{a^2c - a^2 + b^2c + a^2}{a^2 + b^2} - c\Big)v + (a^2 + b^2 - c^2)w \\ &= w, \end{aligned}$$

so that the normal vector to the plane in the (u, v, w)-coordinates is $(0, 0, 1)$. The equation of the cone becomes

$$\Big(\frac{a^2c + b^2}{a^2 + b^2}u + \frac{ab(c-1)}{a^2 + b^2}v + aw\Big)^2 + \Big(\frac{ab(c-1)}{a^2 + b^2}u + \frac{b^2c + a^2}{a^2 + b^2}v + bw\Big)^2 = (-au - bv - cw)^2$$

and the intersection of the cone with the plane $w = d$ is the set of points (u, v) such that

$$\Big(\frac{a^2c + b^2}{a^2 + b^2}u + \frac{ab(c-1)}{a^2 + b^2}v + ad\Big)^2 + \Big(\frac{ab(c-1)}{a^2 + b^2}u + \frac{b^2c + a^2}{a^2 + b^2}v + bd\Big)^2 = (au + bv + cd)^2.$$

Diagonalizing a quadratic form. Let $a, b, c \in \mathbb{R}$ and let

$$\langle , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \qquad \text{be the bilinear form with Gram matrix} \qquad G = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

so that

$$\langle (u_1, u_2), (v_1, v_2) \rangle = \begin{pmatrix} u_1 & u_2 \end{pmatrix} G \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = au_1v_1 + bu_1v_2 + bu_2v_1 + cu_2v_2.$$

If $(x, y) \in \mathbb{R}^2$ then

$$\|(x, y)\|^2 = \langle (x, y), (x, y) \rangle = ax^2 + 2bxy + cy^2.$$

Assuming $b \neq 0$ then

$$G - t = s_1(\frac{a-t}{b}) \begin{pmatrix} b & c-t \\ 0 & b - \frac{(a-t)(c-t)}{b} \end{pmatrix} = s_1(\frac{a-t}{b}) \begin{pmatrix} b & c-t \\ 0 & \frac{-1}{b}(t^2 - (a+c)t - (b^2 - ac)) \end{pmatrix}$$

So $\ker(G-t) \neq 0$ when

$$t = \frac{(a+c) + \sqrt{(a+c)^2 + 4(b^2 - ac)}}{2} \qquad \text{or} \quad t = \frac{(a+c) - \sqrt{(a+c)^2 + 4(b^2 - ac)}}{2}$$

Let $D = \sqrt{a^2 + 2ac + c^2 + 4b^2 - 4ac} = \sqrt{a^2 - 2ac + c^2 + 4b^2} = \sqrt{(a-c)^2 + 4b^2}$ so that the eigenvalues are

$$\lambda_1 = \tfrac{1}{2}(a + c + D) \qquad \text{and} \qquad \lambda_2 = \tfrac{1}{2}(a + c - D).$$

Then

$$\ker(G - \lambda_1) = \mathrm{span}\left\{ \begin{pmatrix} c - \lambda_1 \\ -b \end{pmatrix} \right\} \qquad \text{and} \qquad \ker(G - \lambda_2) = \mathrm{span}\left\{ \begin{pmatrix} c - \lambda_2 \\ -b \end{pmatrix} \right\}.$$

Then

$$\|(c - \lambda_1, -b)\|^2 = a(c - \lambda_1)^2 - 2b^2(c - \lambda_1) + cb^2.$$

Let

$$Q = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \quad \text{where} \quad C = \sqrt{\frac{D + (a-c)}{2D}} \quad \text{and} \quad S = \sqrt{\frac{D - (a-c)}{2D}}.$$

Then

$$Q^T G Q = \begin{pmatrix} \frac{1}{2}(a + c + D) & 0 \\ 0 & \frac{1}{2}(a + c - D) \end{pmatrix}.$$

Remark 2.3. Another way to solve for λ_1 and λ_2 is to use the equations

$$\lambda_1 + \lambda_2 = a + c \qquad \text{and} \qquad \lambda_1 \lambda_2 = ac - b^2. \qquad \square$$

Solutions of a quadratic equation in two variables. The generic quadratic equation in one variable, $ax^2 + bx + c = 0$ with $a, b, c \in \mathbb{R}$, has solutions

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The solutions are in $\mathbb{R}$ exactly when $b^2 - 4ac \geq 0$. The generic quadratic equation in two variables is

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0, \qquad \text{with } a, b, c, d, e, f \in \mathbb{R},$$

and the goal is to find all $(x, y) \in \mathbb{R}^2$ that satisfy this equation.

Let

$$Q = \begin{pmatrix} C & -S \\ S & C \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} z \\ w \end{pmatrix} = Q \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{so that} \quad \begin{matrix} z = Cx + Sy, \\ w = -Sx + Cy, \end{matrix} \quad \text{and} \quad \begin{matrix} x = Cz - Sw, \\ y = Sz + Cw. \end{matrix}$$

Then

$$\begin{aligned}
ax^2+2bxy+cy^2+dx+ey+f &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + f \\
&= \begin{pmatrix} x & y \end{pmatrix} Q^T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} Q \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} Q^T Q \begin{pmatrix} x \\ y \end{pmatrix} + f \\
&= \begin{pmatrix} z & w \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} d & e \end{pmatrix} Q^T \begin{pmatrix} z \\ w \end{pmatrix} + f \\
&= \lambda_1 z^2 + \lambda_2 w^2 + (Cd-Se)z + (Cd+Se)w + f \\
&= \lambda_1\Big(z+\frac{Cd-Se}{2\lambda_1}\Big)^2 + \lambda_2\Big(w+\frac{Cd+Se}{2\lambda_2}\Big)^2 + f - \Big(\frac{Cd-Se}{2\lambda_1}\Big)^2 - \Big(\frac{Cd+Se}{2\lambda_2}\Big)^2 \\
&= \lambda_1(z-A)^2 + \lambda_2(w-B)^2 - F.
\end{aligned}$$

So, after the rotation of the coordinate system specified by Q, the graph of the points where this equation equals 0 is the ellipse or a hyperbola or a parabola given by the equation

$$\lambda_1(z-A)^2 + \lambda_2(w-B)^2 - F = 0.$$