2.8 Week 8 Tutorial: Gram-Schmidt and orthogonal polynomials

The $\mathbb{R}$-vector space of polynomials of degree less than or equal to 4 is

$$\mathbb{R}[x]_{\le 4} = \mathbb{R}\text{span}\{1, x, x^2, x^3, x^4\} = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4\}.$$

Example 1. Legendre polynomials. Define an inner product $\langle , \rangle \colon \mathbb{R}[x]_{\le 4} \times \mathbb{R}[x]_{\le 4} \to \mathbb{R}$ by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

HW: Show that $\langle , \rangle$ is positive definite.

HW: Calculate the Gram matrix of $\langle , \rangle$ with respect to the basis $\{1, x, x^2, x^3, x^4\}$.

HW: Apply the Gram-Schmidt process to $\{1, x, x^2, x^3, x^4\}$ to produce
the orthonormal basis $\{P_0, P_1, P_2, P_3, P_4\}$, where

$$P_0 = 1,$$

$$P_1 = x,$$

$$P_2 = \tfrac{1}{2}(3x^2 - 1),$$

$$P_3 = \tfrac{1}{2}(5x^3 - 3x),$$

$$P_4 = \tfrac{1}{8}(35x^4 - 30x^2 + 3).$$

HW: Show that

$$\begin{aligned}
(1-x^2)\frac{d^2P_0}{dx^2} - 2x\frac{dP_0}{dx} + 0\cdot 1\frac{dP_0}{dx} &= 0,\\
(1-x^2)\frac{d^2P_1}{dx^2} - 2x\frac{dP_1}{dx} + 1\cdot 2\frac{dP_1}{dx} &= 0,\\
(1-x^2)\frac{d^2P_2}{dx^2} - 2x\frac{dP_2}{dx} + 2\cdot 3\frac{dP_2}{dx} &= 0,\\
(1-x^2)\frac{d^2P_3}{dx^2} - 2x\frac{dP_3}{dx} + 3\cdot 4\frac{dP_3}{dx} &= 0, \quad \text{and}\\
(1-x^2)\frac{d^2P_4}{dx^2} - 2x\frac{dP_4}{dx} + 4\cdot 5\frac{dP_4}{dx} &= 0.
\end{aligned}$$

Example 2. Laguerrre polynomials. Define an inner product $\langle , \rangle \colon \mathbb{R}[x]_{\le 4} \times \mathbb{R}[x]_{\le 4} \to \mathbb{R}$ by

$$\langle f, g\rangle = \int_{0}^{\infty} f(x)g(x)e^{-x}\,dx.$$

HW: Show that $\langle , \rangle$ is positive definite.

HW: Calculate the Gram matrix of $\langle , \rangle$ with respect to the basis $\{1, x, x^2, x^3, x^4\}$.

HW: Apply the Gram-Schmidt process to $\{1, x, x^2, x^3, x^4\}$ to produce the orthonormal basis $\{L_0, L_1, L_2, L_3, L_4\}$, where

$$L_0 = 1,$$

$$L_1 = -x + 1,$$

$$L_2 = \tfrac{1}{2}(x^2 - 4x + 2),$$

$$L_3 = \tfrac{1}{6}(-x^3 + 9x^2 - 18x + 6),$$

$$L_4 = \tfrac{1}{24}(x^4 - 16x^3 + 72x^2 - 96x + 24).$$

HW: Show that

$$\begin{aligned}
x\frac{d^2L_0}{dx^2} + (1-x)\frac{dL_0}{dx} + 0\frac{dL_0}{dx} &= 0,\\
x\frac{d^2L_1}{dx^2} + (1-x)\frac{dL_1}{dx} + 1\frac{dL_1}{dx} &= 0,\\
x\frac{d^2L_2}{dx^2} + (1-x)\frac{dL_2}{dx} + 2\frac{dL_2}{dx} &= 0,\\
x\frac{d^2L_3}{dx^2} + (1-x)\frac{dL_3}{dx} + 3\frac{dL_3}{dx} &= 0, \quad \text{and}\\
x\frac{d^2L_4}{dx^2} + (1-x)\frac{dL_4}{dx} + 4\frac{dL_4}{dx} &= 0.
\end{aligned}$$

Example 3. Hermite polynomials. Define an inner product $\langle,\rangle\colon \mathbb{R}[x]_{\le 4} \times \mathbb{R}[x]_{\le 4} \to \mathbb{R}$ by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-x^2}\,dx.$$

HW: Show that $\langle,\rangle$ is positive definite.

HW: Calculate the Gram matrix of $\langle,\rangle$ with respect to the basis $\{1, x, x^2, x^3, x^4\}$.

HW: Apply the Gram-Schmidt process to $\{1, x, x^2, x^3, x^4\}$ to produce the orthonormal basis $\{H_0, H_1, H_2, H_3, H_4\}$, where

$$H_0 = \tfrac{1}{\sqrt{\pi}}1,$$

$$H_1 = \tfrac{1}{2\sqrt{\pi}}2x,$$

$$H_2 = \tfrac{1}{4\cdot 2\sqrt{\pi}}(4x^2 - 2),$$

$$H_3 = \tfrac{1}{8\cdot 6\sqrt{\pi}}(8x^3 - 12x),$$

$$H_4 = \tfrac{1}{16\cdot 24\sqrt{\pi}}16x^4 - 48x^2 + 12).$$

HW: Show that

$$\begin{aligned}
\frac{d^2H_0}{dx^2} - 2x\frac{dL_0}{dx} + 0\frac{dH_0}{dx} &= 0, \\
\frac{d^2H_1}{dx^2} - 2x\frac{dL_1}{dx} + 2\frac{dH_1}{dx} &= 0, \\
\frac{d^2H_2}{dx^2} - 2x\frac{dL_2}{dx} + 4\frac{dH_2}{dx} &= 0, \\
\frac{d^2H_3}{dx^2} - 2x\frac{dL_3}{dx} + 6\frac{dH_3}{dx} &= 0, \quad \text{and} \\
\frac{d^2H_4}{dx^2} - 2x\frac{dL_4}{dx} + 8\frac{dH_4}{dx} &= 0.
\end{aligned}$$