2.6 Week 6 Tutorial: Fibonacci numbers and diagonalisation

The Fibonacci numbers F_n are defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_2 = 1, \quad \text{and} \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \in \mathbb{Z}_{>2}. \qquad \text{(Fibdef)}$$

(a) Use the definition in (Fibdef) to calculate $F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9, F_{10}$.

(b) Define

$$u_n = \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

so that

$$u_{n+1} = \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} F_n \\ F_n + F_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_n \end{pmatrix} = Au_n.$$

Compute $u_1, u_2, \ldots, u_{10}$.

(c) Show that since

$$A - t = \begin{pmatrix} -t & 1 \\ 1 & 1-t \end{pmatrix} = s_1(-t) \begin{pmatrix} 1 & 1-t \\ 0 & 1+t-t^2 \end{pmatrix} = s_1(-t) \begin{pmatrix} 1 & 1-t \\ 0 & -\big(t - \frac{1+\sqrt{5}}{2}\big)\big(1 - \frac{1-\sqrt{5}}{2}\big) \end{pmatrix}$$

then the eigenvectors of A are the columns of the matrix

$$P = \begin{pmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 - \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & \frac{1+\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}.$$

(d) Show that

$$P^{-1} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & \frac{-(1+\sqrt{5})}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = P \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} P^{-1}$$

(f) Show that

$$u_n = A^n u_1 = PD^nP^{-1}u_1, \qquad \text{where} \quad D = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}.$$

Check this formula against your data from (a) for $n \in \{1, \ldots, 9\}$.

(g) Show that the 17th Fibonacci number is

$$F_{17} = \frac{1}{2^{17}\sqrt{5}}(1 - \sqrt{15})^{17}.$$

Marvel at how amazing this formula is.

(h) Use the internet to help you figure out what Fibonacci numbers have to do with rabbits.

(i) Use the internet to help you figure out what Fibonacci numbers have to do with plants like aloe, pinecones and sunflowers. Explain why nature uses these numbers in its biological designs.

(j) Recognise that both (c) and (d) occur via an evolving process (a Markov chain).

(k) Replace the matrix A by the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Using A and the initial conditions $G_1 = 1$, $G_2 = 1$, $G_3 = 1$, what numbers does one get? Do these numbers G_n appear in OEIS (The Online Encyclopedia of Sequences)?