2.5 Week 5 Tutorial: Finding bases – the row space, column space and null space

Let

$$A = \begin{pmatrix} -3 & -36 & -39 & -42 & -46 & -84 \\ -9 & -108 & -\frac{233}{2} & -114 & \frac{293}{2} & -167 \\ 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \end{pmatrix}$$

The *column space of* A is the span of the columns of A,

$$\mathrm{colsp}(A) = \mathrm{span}\left\{ \begin{pmatrix} -3 \\ -9 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -36 \\ -108 \\ 36 \\ 0 \end{pmatrix}, \begin{pmatrix} -39 \\ -\frac{233}{2} \\ 39 \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} -42 \\ -114 \\ 42 \\ 6 \end{pmatrix}, \begin{pmatrix} -46 \\ \frac{293}{2} \\ 45 \\ \frac{25}{4} \end{pmatrix}, \begin{pmatrix} -84 \\ -167 \\ 48 \\ \frac{13}{2}) \end{pmatrix} \right\}$$

The *row space of* A is the span of the rows of B,

$$\mathrm{rowsp}(A) = \mathrm{span}\left\{ \begin{array}{l} (-3, -36, -39, -42, -46, -84), \\ (-9, -108, -\frac{233}{2}, -114, \frac{293}{2}, -167), \\ (3, 36, 39, 42, 45, 48), \\ (0, 0, \frac{1}{4}, 6, \frac{25}{4}, \frac{13}{2} \end{array} \right\}.$$

The *null space of* A is the set

$$\mathrm{nullsp}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \ \Big|\ \begin{pmatrix} -3 & -36 & -39 & -42 & -46 & -84 \\ -9 & -108 & -\frac{233}{2} & -114 & \frac{293}{2} & -167 \\ 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

HW: Show that $\mathrm{im}(A) = \mathrm{colsp}(A)$.

HW: Show that $\ker(A) = \mathrm{nullsp}(A)$.

HW: Let

$$1_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} -3 & -39 & -46 & 1 \\ -9 & -\frac{233}{2} & -\frac{247}{2} & 0 \\ 3 & 39 & 45 & 0 \\ 0 & \frac{1}{4} & \frac{25}{4} & 0 \end{pmatrix} \qquad \text{and} \qquad Q = \begin{pmatrix} 1 & 12 & 0 & -298 & 0 & 10838 \\ 0 & 0 & 1 & 24 & 0 & -874 \\ 0 & 0 & 0 & 0 & 1 & 36 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Verify that $A = P1_rQ$.

HW: Verify that

$$P^{-1} = \begin{pmatrix} 0 & -310 & -\frac{2789}{3} & 568 \\ 0 & 25 & 75 & -46 \\ 0 & -1 & -3 & 2 \\ 1 & -1 & -2 & 2 \end{pmatrix} \qquad \text{and} \qquad Q^{-1} = \begin{pmatrix} 1 & 0 & 0 & -12 & 298 & 10838 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -24 & 874 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -36 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.5.1 Finding a basis of colsp(A)

HW: Show that

$$\mathrm{colsp}(A) \subseteq \mathrm{colsp}(P1_3) \qquad \text{and} \qquad \mathrm{colsp}(P1_3) \subseteq \mathrm{colsp}(A).$$

HW: Let $P\mathrm{colsp}(1_3) = \{Py \mid y \in \mathrm{colsp}(1_3)\}$. Show that

$$P\mathrm{colsp}(1_3) \subseteq \mathrm{colsp}(P1_3) \qquad \text{and} \qquad \mathrm{colsp}(P1_3) \subseteq P\mathrm{colsp}(1_3).$$

HW: Show that

$$\mathrm{span}\left\{ \begin{pmatrix} -3 \\ -9 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -39 \\ -\frac{233}{2} \\ 39 \\ \frac{1}{4} \end{pmatrix}, \begin{pmatrix} -46 \\ -\frac{247}{2} \\ 45 \\ \frac{25}{4} \end{pmatrix} \right\} \qquad \text{is a basis of colsp}(A).$$

(Yes, a necessary step is to show that the 3 vectors are linearly independent.) Conclude that the first 3 columns of P form a basis of colsp(A).

2.5.2 Finding a basis of nullsp(A)

HW: Show that

$$\mathrm{nullsp}(A) \subseteq \mathrm{nullsp}(1_3Q) \qquad \text{and} \qquad \mathrm{nullsp}(1_3Q) \subseteq \mathrm{nullsp}(A).$$

HW: Let $Q^{-1}\mathrm{nullsp}(1_3) = \{Q^{-1}y \mid y \in \mathrm{nullsp}(1_3)\}$. Show that

$$Q^{-1}\mathrm{nullsp}(1_3) \subseteq \mathrm{nullsp}(1_3Q) \qquad \text{and} \qquad \mathrm{nullsp}(1_3Q) \subseteq Q^{-1}\mathrm{nullsp}(1_3).$$

HW: Show that

$$\mathrm{span}\left\{ \begin{pmatrix} -12 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 298 \\ 0 \\ -24 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 10838 \\ 0 \\ 874 \\ 0 \\ -36 \\ 1 \end{pmatrix} \right\} \qquad \text{is a basis of nullsp}(A).$$

(Yes, a necessary step is to show that the 3 vectors are linearly independent.) Conclude that the last 3 columns of Q^{-1} form a basis of nullsp(A).

2.5.3 Finding a basis of rowsp(A)

HW: Show that

$$\mathrm{rowsp}(A) \subseteq \mathrm{colsp}(1_3Q) \qquad \text{and} \qquad \mathrm{rowsp}(1_3Q) \subseteq \mathrm{rowsp}(A).$$

HW: Let $\mathrm{rowsp}(1_3)Q = \{yQ \mid y \in \mathrm{rowsp}(1_3)\}$. Show that

$$\mathrm{rowsp}(1_3)Q \subseteq \mathrm{rowsp}(1_3Q) \qquad \text{and} \qquad \mathrm{rowsp}(1_3Q) \subseteq \mathrm{rowsp}(1_3)Q.$$

HW: Show that

$$\mathrm{span}\{(1, 12, 0, -298, 0, 10838), (0, 0, 1, 24, 0, -874), (0, 0, 0, 0, 1, 36)\} \qquad \text{is a basis of colsp}(A).$$

(Yes, a necessary step is to show that the 3 vectors are linearly independent.) Conclude that the nonzero rows of 1_3Q form a basis of rowsp(A).