2.4 Week 4 Tutorial: Solving problems with an unknown parameter.

Example L11. Find the values of $a, b \in \mathbb{Q}$ for which the system

$$\begin{array}{rcl} x - 2y + z = 4, & & \text{(a) no solution,} \\ 2x - 3y + z = 7, & \text{has} & \text{(b) a unique solution,} \\ 3x - 6y + az = b, & & \text{(c) LOTS of solutions.} \end{array}$$

In matrix form this system is

$$\begin{pmatrix} 3 & -6 & a \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 7 \\ 4 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 3 & -6 & a \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 4 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & a-3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ b-12 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & a-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ b-12 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & a-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ b-12 \end{pmatrix}.$$

Case 1: $a - 3 \neq 0$. Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a-3} \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ \frac{b-12}{a-3} \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 + \frac{b-12}{a-3} \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix}.$$

So

$$\begin{aligned} x &= 2 + \tfrac{b-12}{a-3}, \\ y &= -1 + \tfrac{b-12}{a-3}, \\ z &= \tfrac{b-12}{a-3}, \end{aligned}$$

or, equivalently,

$$\mathrm{Sol}(Ax=b) = \left\{ \begin{pmatrix} 2 + \frac{b-12}{a-3} \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix} \right\} \qquad \text{(exactly \textit{one} solution).}$$

Case 2: $a-3=0$. Then

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ b-12 \end{pmatrix}.$$

Case 2a: $b-12 \neq 0$. If $b-12 \neq 0$ then this system has *no solution.*

Case 2b: $b-12=0$. If $b-12=0$ then this system is

$$\begin{aligned} x - z &= 2, \\ y - z &= -1, \end{aligned} \qquad \text{which is} \qquad \begin{aligned} x &= 2+z, \\ y &= -1+z, \\ z &= 0+z, \end{aligned}$$

where z can be any number. So

$$\mathrm{Sol}(Ax=b) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mathrm{span}\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and there are *LOTS of solutions.*