2.3 Week 3 Tutorial: Inverse and normal form for generic 2×2 and 3×3 matrices

HW: Let $a,b,c,d\in\mathbb{Q}$. Let

$$A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\qquad\text{and}\qquad B=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}.$$

Compute AB and BA.

HW: Let $a,b,c,d\in\mathbb{Q}$. Prove that if $ad-bc\neq 0$ then

$$A^{-1}=\begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{pmatrix}.$$

HW: Let $a_{11},a_{12},a_{13},a_{21},a_{22},a_{23},a_{31},a_{32},a_{33}\in\mathbb{Q}$. Let $A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$ and

$$B=\begin{pmatrix} a_{22}a_{33}-a_{23}a_{32} & -(a_{12}a_{33}-a_{33}a_{32}) & a_{12}a_{23}-a_{13}a_{22}\\ -(a_{21}a_{23}-a_{23}a_{31}) & a_{11}a_{33}-a_{13}a_{31} & -(a_{11}a_{23}-a_{13}a_{21})\\ a_{21}a_{32}-a_{22}a_{31} & -(a_{11}a_{32}-a_{12}a_{31}) & a_{11}a_{22}-a_{12}a_{21}\end{pmatrix}.$$

Compute AB and BA.

HW: Let $a_{11},a_{12},a_{13},a_{21},a_{22},a_{23},a_{31},a_{32},a_{33}\in\mathbb{Q}$ and let $A=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$. Let

$$D=a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{12}a_{21}a_{33}-a_{11}a_{23}a_{32}-a_{13}a_{22}a_{21}.$$

Prove that if $D\neq 0$ then

$$A^{-1}=\begin{pmatrix} \frac{a_{22}a_{33}-a_{23}a_{32}}{D} & \frac{-(a_{12}a_{33}-a_{33}a_{32})}{D} & \frac{a_{12}a_{23}-a_{13}a_{22}}{D}\\ \frac{-(a_{21}a_{23}-a_{23}a_{31})}{D} & \frac{a_{11}a_{33}-a_{13}a_{31}}{D} & \frac{-(a_{11}a_{23}-a_{13}a_{21})}{D}\\ \frac{a_{21}a_{32}-a_{22}a_{31}}{D} & \frac{-(a_{11}a_{32}-a_{12}a_{31})}{D} & \frac{a_{11}a_{22}-a_{12}a_{21}}{D}\end{pmatrix}.$$

Example A1. Let $a,b,c,d\in\mathbb{Q}$ and find the inverse of $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$.

Start with $AA^{-1}=1$, which is

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}A^{-1}=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}.$$

Case 1: $c\neq 0$. Left multiply by $s_1(\frac{a}{c})^{-1}$, which is the matrix

$$\begin{pmatrix} 0 & 1\\ 1 & -\frac{a}{c}\end{pmatrix},\quad\text{to get}\begin{pmatrix} c & d\\ 0 & b-\frac{a}{c}d\end{pmatrix}A^{-1}=\begin{pmatrix} 0 & 1\\ 1 & -\frac{a}{c}\end{pmatrix}.$$

Case 1a: $c\neq 0$ and $ad-bc\neq 0$. Left multiply by $h_1(c)^{-1}h_2(\frac{bc-ad}{c})^{-1}$, which is the matrix

$$\begin{pmatrix} \frac{1}{c} & 0\\ 0 & \frac{c}{bc-ad}\end{pmatrix},\quad\text{to get}\begin{pmatrix} 1 & \frac{d}{c}\\ 0 & 1\end{pmatrix}A^{-1}=\begin{pmatrix} 0 & \frac{1}{c}\\ \frac{c}{bc-ad} & -\frac{a}{bc-ad}\end{pmatrix}.$$

Left multiply by $x_{12}(\frac{d}{c})^{-1}$, which is the matrix

$$\begin{pmatrix} 1 & -\frac{d}{c} \\ 0 & 1 \end{pmatrix}, \quad \text{to get} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} -\frac{d}{bc-ad} & \frac{1}{c} + \frac{ad}{c(bc-ad)} \\ \frac{c}{bc-ad} & -\frac{a}{bc-ad} \end{pmatrix}.$$

So

$$A^{-1} = \begin{pmatrix} -\frac{d}{bc-ad} & \frac{bc-ad+ad}{c(bc-ad)} \\ \frac{c}{bc-ad} & -\frac{a}{bc-ad} \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Case 1b: $c \neq 0$ and $ad - bc = 0$. Then

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{a}{c} \end{pmatrix}$$

and **there does not exist any matrix A^{-1} that makes this equation true** in this case.

Case 2: $c = 0$. Then

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Case 2a: $c = 0$ and $d \neq 0$ and $a \neq 0$ so that $ad - bc = ad - b \cdot 0 \neq 0$. Left multiply by $h_2(d)^{-1}h_1(a)^{-1}$, which is the matrix

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}.$$

Left multiply by $x_{12}(\frac{b}{a})^{-1}$, which is the matrix

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{ad} \\ 0 & \frac{1}{d} \end{pmatrix} = \begin{pmatrix} \frac{d}{ad} & \frac{-b}{ad} \\ 0 & \frac{a}{ad} \end{pmatrix}.$$

Recalling that $c = 0$ then

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Case 2b: $c = 0$ and $a \neq 0$. Then

$$\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and **there does not exist any matrix A^{-1} that makes this equation true** in this case. *Case 2c: $c = 0$ and $d \neq 0$.* Then

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and **there does not exist any matrix A^{-1} that makes this equation true** in this case.

Altogether, this thorough consideration of the various cases has proved (by row reduction) the following theorem.

Theorem 2.1. ***(Inverse of a* 2×2 *matrix)*** *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Q})$. *Then*

(a) *If $ad - bc \neq 0$ then* $A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

(b) *If $ad - bc = 0$ then A^{-1} does not exist.*

Example M5. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$. Then

$$A^{-1} = \frac{1}{(2\cdot 1 - 1\cdot(-1))}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{3}{3} & 0 \\ 0 & \frac{3}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

HW: Use row reduction (in analogy with Example A1) to prove the following theorem.

Theorem 2.2. ***(Inverse of a*** 3×3 ***matrix)*** *Let* $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33} \in \mathbb{Q}$ *and let*

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Let

$$D = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{21} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31}.$$

(a) If $D \neq 0$ *then*

$$A^{-1} = \begin{pmatrix} \frac{a_{22}a_{33} - a_{23}a_{32}}{D} & \frac{-(a_{12}a_{33} - a_{33}a_{32})}{D} & \frac{a_{12}a_{23} - a_{13}a_{22}}{D} \\ \frac{-(a_{21}a_{23} - a_{23}a_{31})}{D} & \frac{a_{11}a_{33} - a_{13}a_{31}}{D} & \frac{-(a_{11}a_{23} - a_{13}a_{21})}{D} \\ \frac{a_{21}a_{32} - a_{22}a_{31}}{D} & \frac{-(a_{11}a_{32} - a_{12}a_{31})}{D} & \frac{a_{11}a_{22} - a_{12}a_{21}}{D} \end{pmatrix}.$$

(b) If $D = 0$ *then* A^{-1} *does not exist.*