2.2 Week 2 Tutorial: Generator matrices and row and column operations

Let $n \in \mathbb{Z}_{>0}$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i,j) entry and 0 elsewhere.

Root matrices, diagonal generators and row reducers.

Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Let $c \in \mathbb{Q}$. The *root matrix* $x_{ij}(c)$ is

$$x_{ij}(c) \in M_{n\times n}(\mathbb{Q}) \quad \text{given by} \quad x_{ij}(c) = 1 + cE_{ij}.$$

Let $i \in \{1, \ldots, n\}$. Let $d \in \mathbb{Q}$ with $d \neq 0$. The *diagonal generator* $h_i(d)$ is

$$h_i(d) = 1 + (d-1)E_{ii}.$$

Let $i \in \{1, \ldots, n-1\}$ and let $c \in \mathbb{Q}$. The *row reducer* $s_i(c)$ is

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

Row operations and root matrices

Let

$$A = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \quad \text{and} \quad x_{13}(54) = \begin{pmatrix} 1 & 0 & 54 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Left multiplying by $x_{13}(54)$ adds $54 \cdot (\text{row } 3)$ to row 1:

$$x_{13}(54)A = \begin{pmatrix} 1 & 0 & 54 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} = \begin{pmatrix} 16203 & -5409 & 10807 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix}.$$

Row operations for diagonal generators

Let

$$A = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \quad \text{and} \quad h_3(6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Left multiplying by $h_3(6)$ multiplies row 3 by 6:

$$h_3(6)A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 1800 & -600 & 1200 \end{pmatrix}.$$

Row operations for row reducers

Let

$$A = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \quad \text{and} \quad s_2(-5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Left multiplying by $s_2(-5)$ moves row 2 to be row 3 and makes row 2 equal to $(-5)\cdot(\text{row } 2) + (\text{row } 3)$:

$$s_2(-5) \cdot A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} = \begin{pmatrix} 3 & -9 & 7 \\ 235 & 5 & 25 \\ 13 & -21 & 35 \end{pmatrix}.$$

Column operations and root matrices
Let

$$A = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \quad \text{and} \quad x_{13}(54) = \begin{pmatrix} 1 & 0 & 54 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Right multiplying by $x_{13}(54)$ adds $54 \cdot$ (column 1) to column 3:

$$A \cdot x_{13}(54) = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \begin{pmatrix} 1 & 0 & 54 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -9 & 169 \\ 13 & -21 & 737 \\ 300 & -100 & 16400 \end{pmatrix}.$$

Column operations for diagonal generators
Let

$$A = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \quad \text{and} \quad h_3(6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Right multiplying by $h_3(6)$ multiplies column 3 by 6:

$$A \cdot h_3(6) = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 3 & -9 & 42 \\ 13 & -21 & 210 \\ 300 & -100 & 1200 \end{pmatrix}.$$

Column operations for row reducers
Let

$$A = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \quad \text{and} \quad s_2(-5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Right multiplying by $s_2(-5)$ moves column 2 to be column 3 and makes column 2 equal to $(-5) \cdot$ (column 2) + (column 3):

$$A \cdot s_2(-5) = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 52 & -9 \\ 13 & 140 & -21 \\ 300 & 700 & -100 \end{pmatrix}.$$

HW: Check that

$$s_1(c_1)s_2(c_2)s_1(c_3) = s_2(c_3)s_1(c_2 - c_1c_3)s_2(c_1)$$

by computing

$$\begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_3 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 - c_1c_3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$