2 Tutorials

2.1 Week 1 Tutorial: 3d-space-time and cross products

The 3*d-space-time*, or the *Hamiltonians*, or the *quaternions*, is the vector space

$$\mathbb{D} = \mathbb{R}\text{-span}\{1, i, j, k\} = \{t + xi + yj + zk \mid t, x, y, z \in \mathbb{R}\}$$

with product determined by

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j,$$

and the distributive laws. The 3d-space-time $\mathbb{D}$ is a generalization of 1d-space-time $\mathbb{C}$, where

$$\mathbb{C} = \{xi + t \mid x, t \in \mathbb{R}\} \qquad \text{with product determined by} \quad i^2 = -1 \quad \text{and the distributive laws.}$$

HW: Compute $(3 + 2i + 4j - 6k) + (-4 + 7i - 4j + 2k)$ and $7 \cdot (4 + 2i + 6j - 9k)$.

HW: Compute $(3+2i)\cdot(3-2i)$ and $(3+2j)\cdot(3-2j)$ and $(3+2i)\cdot(3-2k)$ and $(3+2i)\cdot(3-2j)\cdot(1-k)$.

HW: Compute $(3 + 2i + 4j - 6k) \cdot (-4 + 7i - 4j + 2k)$ and $(-4 + 7i - 4j + 2k) \cdot (3 + 2i + 4j - 6k)$.

The 3*d-space* $\mathbb{R}^3$ is a subspace of 3d-space-time $\mathbb{D}$,

$$\begin{array}{ll} \mathbb{D} & = \{t + xi + yj + zk \mid t, x, y, z \in \mathbb{R}\} \\ \cup| & \\ \mathbb{R}^3 & = \{xi + yj + zk \mid x, y, z \in \mathbb{R}\} \end{array}$$

For $v_1 = x_1 i + y_1 j + z_1 k$ and $v_2 = x_2 i + y_2 j + z_2 k$ in $\mathbb{R}^3$ define

$$\begin{array}{l} \langle v_1 \mid v_2 \rangle = x_1 y_1 + y_1 y_2 + z_1 z_2 \qquad \text{and} \\ v_1 \times v_2 = (y_1 z_2 - z_1 y_2)i - (x_1 z_2 - z_1 x_2)j + (x_1 y_2 - y_1 x_2)k. \end{array}$$

HW: Let $v_1 = 2i + j - 3k$ and $v_1 = -5i + 2j - k$. Compute $\langle v_1 | v_2 \rangle$ and $v_1 \times v_2$.

HW: Let $v_1 = 5i + 12j + 13k$ and $v_2 = 5i + 12j - 13k$. Compute $\langle v_1 | v_2 \rangle$ and $v_1 \times v_2$.

HW: Compute

$$\begin{array}{ccc} \langle i|i\rangle, & \langle i|j\rangle, & \langle i|k\rangle, \\ \langle j|i\rangle, & \langle j|j\rangle, & \langle j|k\rangle, \\ \langle k|i\rangle, & \langle k|j\rangle, & \langle k|k\rangle, \\ i \times i, & i \times j, & i \times k, \\ j \times i, & j \times j, & j \times k, \\ k \times i, & k \times j, & k \times k. \end{array}$$

HW: Prove that if $v_1 = x_1 i + y_1 j + z_1 k$ and $v_2 = x_2 i + y_2 j + z_2 k$ in $\mathbb{R}^3$ then

$$v_1 v_2 = -\langle v_1 \mid v_2 \rangle + v_1 \times v_2.$$

This computation shows how the standard inner product and the cross product arise from the multiplication in 3d-space-time.

The *Galois automorphism*, or *conjugation map*, is the function $\overline{}\colon \mathbb{D} \to \mathbb{D}$ given by

$$\overline{t+xi+yj+zk} = t - xi - yj - zk, \qquad \text{for } t,x,y,z \in \mathbb{R}.$$

The *norm on* $\mathbb{D}$ is the function $\|\ \|\colon \mathbb{D} \to \mathbb{R}_{\ge 0}$ given by

$$\|d\| = \sqrt{d\,\overline{d}}.$$

HW:. Show that if $d_1, d_2 \in \mathbb{D}$ then $\overline{d_1 d_2} = \overline{d_1} \cdot \overline{d_2}$.
HW:. Show that if $t,x,y,z \in \mathbb{R}$ and $d = t+xi+yj+zk$ then

$$d\overline{d} = t^2 + x^2 + y^2 + z^2.$$

HW:. Show that if $d \in \mathbb{D}$ and $d \neq 0$ then

$$\frac{1}{\|d\|^2}\overline{d} = d^{-1}.$$

HW:. Let $d_1 = 1 + 2i - 3j - 2k$ and $d_2 = 4 - 2i + 5j$ Compute

$$d_1 \cdot d_2 \qquad \text{and} \qquad \overline{d_1 \cdot d_2}$$

and then compute

$$\overline{d_1}, \qquad \overline{d_2}, \qquad \text{and} \qquad \overline{d_1} \cdot \overline{d_2}.$$

HW:. Let $d_1 = 1 + 2i - 3j - 2k$ and $d_2 = 4 - 2i + 5j$ Compute

$$d_1\overline{d_1}, \qquad \|d_1\|, \qquad d_1\overline{d_2}, \qquad \|d_2\|.$$

HW:. Let $d_1 = 1 + 2i - 3j - 2k$ and $d_2 = 4 - 2i + 5j$ and let

$$f_1 = \frac{1}{\|d_1|}\overline{d_1} \qquad \text{and} \qquad f_2 = \frac{1}{\|d_2|}\overline{d_2}.$$

Compute

$$f_1, \qquad f_2, \qquad f_1 \cdot d_1, \qquad d_1 \cdot f_1, \qquad f_2 \cdot d_2, \qquad \text{and} \qquad d_2 \cdot f_2.$$

HW: The *(normalized) Pauli matrices* are the matrices $I, J, K \in M_{2\times 2}(\mathbb{C})$ given by

$$I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Show that

$$I^2 = J^2 = K^2 = -1, \qquad IJ = -JI = K, \qquad JK = -KJ = I, \qquad KI = -IK = J.$$