2.12 Week 12 Tutorial: A past exam

Do the following past exam. Have another student in the class mark it. Does your performance on this match your mark expectation?

Question 1 (9 marks)

(a) Assume that
$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{and} \quad \det(M) = 1.$$

(i) Evaluate $\det \begin{pmatrix} d & e & f \\ a & b & c \\ g-4a & h-4b & i-4c \end{pmatrix}$.

(ii) Evaluate $\det \begin{pmatrix} 3a & 3b & 3c \\ 3d & 3e & 3f \\ 3g & 3h & 3i \end{pmatrix}$.

(iii) Is M invertible? Explain your answer.

(b) Find the equation of the polynomial $y = ax^2+bx+c$ that passes through the points $(1,2), (-1,6)$ and $(2,3)$ as follows:

(i) Write down a system of linear equations for a, b and c.

(ii) Solve the linear system using row operations. Indicate the row operations used in each step.

Question 2 (8 marks) Let $\mathbf{u}$ be a non-zero vector in $\mathbb{R}^3$. Consider the function $t\colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by
$$T(\mathbf{x}) = \mathbf{u} \times \mathbf{x},$$
where $\mathbf{u} \times \mathbf{x}$ is the cross product.

(a) Prove that T is a linear transformation.

(b) Find the kernel and image of T for a general $\mathbf{u} = (u_1, u_2, u_3)$.

Question 3 (7 marks) Let $O = (0,0,0)$,A=(2,0,1), $B = (2,1,5)$, $C = (1,3,5)$ be points in $\mathbb{R}^3$.

(a) Find the area of the triangle with vertices A, B and C.

(b) Let ℓ_1 be the line passing through the points A and C. Find the Cartesian form of the line which is parallel to ℓ_1 and passes through B.

Question 4 (13 marks)

(a) Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the vector space of real-valued functions on $\mathbb{R}$. Determine, with proof, if the following sets are subspaces.

(i) The set S of all functions $\mathbf{f}\colon \mathbb{R} \to \mathbb{R}$ such that $\mathbf{f}(x) = f(7-x)$ for all $x \in \mathbb{R}$.

(ii) The set W of all functions $\mathbf{f}\colon \mathbb{R} \to \mathbb{R}$ such that $\mathbf{f}(x) \leq 1$ for all $x \in \mathbb{R}$.

(b) Determine whether $\{(1,1,1),(i,0,1),(0-i,i)$ is a basis for $\mathbb{C}^3$.

Question 5 (13 marks) Let $M_{2\times 2}(\mathbb{R})$ be the real vector space of all 2×2 matrices with entries real numbers. Let $\mathcal{B}$ be the ordered basis of $M_{2\times 2}(\mathbb{R})$ given by

$$\mathcal{B} = \{E_{11}, E_{12}, E_{21}, E_{22}\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, , \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation defined by

$$T(A) = PA - AQ, \qquad \text{where} \quad P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

(a) What is $T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$?

(b) Find the matrix representation of T with respect to the basis $\mathcal{B}$.

(c) Compute the rank of T.

(d) Is T surjective? Explain your answer.

(e) Is T injective? Explain your answer.

Question 6 (10 marks) For each of the following, give an example or explain why no such exists.

(a) A surjective linear transformation $T\colon \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$.

(b) Let $\mathcal{S} = \{(1,0),(0,1)\}$ be the standard basis of $\mathbb{R}^2$. A linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which is not injective and each row of $[T]_{\mathcal{S},\mathcal{S}}$ is not zero.

(c) A symmetric matrix $A \in M_{2\times 2}(\mathbb{R})$ such that $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $A \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with both $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ being eigenvectors of A.

(d) Recall that a matrix $A \in M_n(\mathbb{R})$ is said to be *diagonalisable* if there exists an invertible matrix P such that

$$P^{-1}AP$$

is a diagonal matrix. A matrix $A \in M_n(\mathbb{R})$ which is invertible but not diagonalisable.

(e) A non-empty subset of $\mathbb{R}^2$ that is closed under addition and taking inverses but not a subspace.

Question 7 (10 marks) Consider the ordered set

$$\mathcal{B} = \{(0,1,0),(1,1,1),(0,0,1)\}$$

and the ordered standard basis

$$\mathcal{S} = \{(1,0,0),(0,1,0),(0,0,1)\}$$

for $\mathbb{R}^3$.

(a) Verify that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

(b) Write down the transition matrix $[I]_{\mathcal{S},\mathcal{B}}$.

(c) Calculate the transition matrix $[I]_{\mathcal{B},\mathcal{S}}$.

(d) You are given that for a particular linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$,

$$[T]_{\mathcal{S},\mathcal{S}} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{pmatrix}.$$

Calculate $[T]_{\mathcal{B},\mathcal{B}}$.

(e) Calculate $[T(1,-1,1)]_{\mathcal{B}}$.

Question 8 (10 marks) Consider the matrix

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) Find the eigenvectors of A.

(b) Find a basis for the eigenspace of A corresponding to each eigenvalue.

(c) For $u, v \in M_{3\times 1}(\mathbb{R})$ define

$$\langle u, v \rangle = u^T A v.$$

Decide (with explanation) whether or not this defines an inner product on $M_{3\times 1}(\mathbb{R})$.

Question 9 (10 marks) Let $\mathbf{u} = (u_1, u_2) \in \mathbb{C}^2$, $\mathbf{v} = (v_1, v_2) \in \mathbb{C}^2$ and

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1\overline{v_1} + 2u_2\overline{v_2},$$

where $\overline{v_1}$ is the complex conjugate of v_1.

(a) Show that $\langle \mathbf{u}, \mathbf{v} \rangle$ defines a Hermitian inner product on the complex vector space $\mathbb{C}^2$.

(b) Suppose that $\mathbf{u} = (1+3i, -2i)$ and $\mathbf{v} = (-2, 2-i)$. Using this inner product, find the

(i) distance between $\mathbf{u}$ and $\mathbf{v}$;

(i) angle between $\mathbf{u}$ and $\mathbf{v}$.

Question 10 (10 marks) Consider the vector space $V = \mathbb{R}[x]_{\le 2}$ of polynomials of degree less than or equal to 2. Let $\mathbf{f}, \mathbf{g} \in \mathbb{R}[x]_{\le 2}$. Define their inner product to be

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_0^1 f(x)g(x)\,dx.$$

(a) Let W be the subspace of V spanned by

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \{x, x^2 - x\}.$$

Use the Gram-Schmidt procedure to find an orthonormal basis for W.

(b) Find the point in W that is closest to $1 \in V$.