2.11 Week 11 Tutorial: Comparing formulas for the determinant

2.11.1 Determinants by factoring

Let $n \in \mathbb{Z}_{>0}$. Let E_{ij} be the $n\times n$ matrix with 1 in the (i,j) entry and 0 elsewhere. For $i \in \{1,\dots,n-1\}$ and $c \in \mathbb{Q}$ define

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

For $i \in \{1,\dots,n\}$ and $d \in \mathbb{Q}$ with $d \neq 0$ define

$$h_i(d) = 1 + (d-1)E_{ii},$$

For $i,j \in \{1,\dots,n\}$ with $i \neq j$ and $c \in \mathbb{Q}$ define

$$x_{ij}(c) = 1 + cE_{ij},$$

For $r \in \{1,\dots,n\}$ define

$$1_r = E_{11} + \cdots + E_{rr}.$$

The *deteminant* is the function $\det\colon M_{n\times n}(\mathbb{Q}) \to \mathbb{Q}$ determined by

$$\text{if } A,B \in M_{n\times n}(\mathbb{Q}) \text{ then } \quad \det(AB) = \det(A)\det(B),$$

and if $i,j \in \{1,\dots,n\}$ with $i \neq j$ and $k \in \{1,\dots,n-1\}$ and $c,d \in \mathbb{Q}$ with $d \neq 0$ then

$$\det(x_{ij}(c)) = 1, \quad \det(h_i(d)) = d, \quad \det(s_k(c)) = -1.$$

Theorem 2.6. ***(Factoring)*** *Let $n \in \mathbb{Z}_{>0}$. Let*

$$1_r = E_{11} + \cdots + E_{rr} \qquad \text{in } M_{n\times n}(\mathbb{Q}).$$

Let $A \in M_{n\times n}(\mathbb{Q})$. The factoring algorithm gives

$$\begin{aligned} A = (\text{product of } s_i(c)s) \cdot (\text{product of } h_i(d)s) \cdot (\text{product of } x_{ij}(c)s) \\ \cdot 1_r \cdot (\text{product of } s_i(c)s) \cdot (\text{product of } x_{ij}(c)s). \end{aligned}$$

Example M12. Let $A = \begin{pmatrix} 2 & 6 & 9 \\ 0 & 3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$. Then

$$A = h_1(2)h_2(3)h_3(-1)\begin{pmatrix} 1 & 3 & \frac{9}{2} \\ 0 & 1 & \frac{8}{3} \\ 0 & 0 & 1 \end{pmatrix} = h_1(2)h_2(3)h_3(-1)x_{12}(3)x_{13}(\tfrac{9}{2})x_{23}(\tfrac{8}{3})$$

and

$$\begin{aligned} \det(A) &= \det\big(h_1(2)h_2(3)h_3(-1)x_{12}(3))x_{13}(\tfrac{9}{2})x_{23}(\tfrac{8}{3})\big) \\ &= \det(h_1(2))\det(h_2(3))\det(h_3(-1))\det(x_{12}(3)))\det(x_{13}(\tfrac{9}{2}))\det(x_{23}(\tfrac{8}{3})) \\ &= 2\cdot 3\cdot(-1)\cdot 1\cdot 1\cdot 1 = -6. \end{aligned}$$

$\square$

Example M13. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Then

$$\begin{aligned} A &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{pmatrix} \\ &= s_1(-1)s_2(3) \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{pmatrix} = s_1(-1)s_2(3)h_1(-1)h_2(1)h_3(-7) \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \\ &= s_1(-1)s_2(3)h_1(-1)h_2(1)h_3(-7)x_{12}(-1)x_{13}(-1)x_{23}(3) \end{aligned}$$

and $\det(A) = (-1)\cdot(-1)\cdot(-1)\cdot 1\cdot(-7)\cdot 1\cdot 1\cdot 1 = 7$. □

2.11.2 The permutation formula for the determinant

Let $n \in \mathbb{Z}_{>0}$. A *permutation* is an $n\times n$ matrix such that each row and each column contain exactly one 1 and all other entries are 0. If $n = 2$ then the permtutations are the matrices in the set

$$S_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

If $n = 3$ then the permutations are the matrices in the set

$$S_3 = \left\{ \begin{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, & \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{matrix} \right\}.$$

Let $w(i)$ be the row number of the 1 in column j of w. Let

$$\mathrm{Inv}(w) = \{(i,j) \mid i,j \in \{1,\ldots,n\} \text{ and } i<j \text{ and } w(i)>w(j)\} \quad \text{and let} \quad \ell(w) = \#\mathrm{Inv}(w).$$

The permutation for the determinant of a matrix A is

$$\det(A) = \sum_{w\in S_n} (-1)^{\ell(w)} A(1,w(1))A(2,w(2))\cdots A(n,w(n)).$$

If $n = 2$ the permutation formula says

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \underset{\begin{pmatrix} \color{red}{a_{11}} & a_{12} \\ a_{21} & \color{red}{a_{22}} \end{pmatrix}}{(-1)^0 a_{11}a_{12}} + \underset{\begin{pmatrix} a_{11} & \color{red}{a_{12}} \\ \color{red}{a_{21}} & a_{22} \end{pmatrix}}{(-1)^1 a_{12}a_{21}} = a_{11}a_{22} - a_{12}a_{21}.$$

If $n = 3$ the permutation formula says

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = (-1)^0 a_{11}a_{22}a_{33} + (-1)^2 a_{12}a_{23}a_{31} + (-1)^2 a_{13}a_{21}a_{32}$$
$$+(-1)^1 a_{12}a_{21}a_{33} + (-1)^1 a_{11}a_{23}a_{32} + (-1)^3 a_{13}a_{22}a_{31}.$$

corersponding to

$$\begin{pmatrix} \color{red}{a_{11}} & a_{12} & a_{13} \\ a_{21} & \color{red}{a_{22}} & a_{23} \\ a_{31} & a_{32} & \color{red}{a_{33}}\end{pmatrix} \quad \begin{pmatrix} a_{11} & \color{red}{a_{12}} & a_{13} \\ a_{21} & a_{22} & \color{red}{a_{23}} \\ \color{red}{a_{31}} & a_{32} & a_{33}\end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & \color{red}{a_{13}} \\ \color{red}{a_{21}} & a_{22} & a_{23} \\ a_{31} & \color{red}{a_{32}} & a_{33}\end{pmatrix}$$

$$\begin{pmatrix} a_{11} & \color{red}{a_{12}} & a_{13} \\ \color{red}{a_{21}} & a_{22} & a_{23} \\ a_{31} & a_{32} & \color{red}{a_{33}}\end{pmatrix} \quad \begin{pmatrix} \color{red}{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & \color{red}{a_{23}} \\ a_{31} & \color{red}{a_{32}} & a_{33}\end{pmatrix} \quad \begin{pmatrix} a_{11} & a_{12} & \color{red}{a_{13}} \\ a_{21} & \color{red}{a_{22}} & a_{23} \\ \color{red}{a_{31}} & a_{32} & a_{33}\end{pmatrix}$$

So

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{pmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}.$$

2.11.3 Cofactor formulas for the determinant

Let $A \in M_{n\times n}(\mathbb{Q})$ and let $i, j \in \{1, \ldots, n\}$.

Let $A^{(i,j)}$ be the matrix A with the ith row removed and the jth column removed.

The *(i, j)-cofactor of A* is

$$C_{ij} = (-1)^{i+j}\det(A^{(i,j)}).$$

Theorem 2.7. ***(cofactor expansion)*** *Let $A \in M_{n\times n}(\mathbb{Q})$ and let $i, j \in \{1, \ldots, n\}$. Then*

$$\det(A) = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in} \qquad \textit{cofactor expansion across the } i\textit{th row}$$

and

$$\det(A) = A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj}. \qquad \textit{cofactor expansion down the } j\textit{th column}$$

Example M15 and 16. If $A = \begin{pmatrix} \color{blue}{1} & \color{blue}{2} & 1 \\ -1 & 1 & \color{red}{1} \\ \color{blue}{0} & \color{blue}{1} & 3\end{pmatrix}$ then the (2,3)-cofactor is

$$C_{23} = (-1)^{2+3}\det\begin{pmatrix} \color{blue}{1} & \color{blue}{2} \\ \color{blue}{0} & \color{blue}{1}\end{pmatrix} = (-1)^5(1\cdot 1 - 0\cdot 2) = -1.$$

Using cofactor expansion along the third row,

$$\begin{aligned}\det(A) &= (-1)^{3+1}\cdot 0\cdot\det\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix} + (-1)^{3+2}\cdot 1\cdot\det\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix} \\ &\quad + (-1)^{3+3}\cdot 3\cdot\det\begin{pmatrix} 1 & 2 \\ -1 & 1\end{pmatrix} \\ &= 0 - (1\cdot 1 - (-1)\cdot 1) + 3(1\cdot 1 - (-1)\cdot 2) \\ &= 0 - 2 + 9 = 7.\end{aligned}$$

□

Example M17. Let $A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 4 \end{pmatrix}$.

(a) Calculate the determinant of A by factoring A as a product of elementary matrices.

(b) Calculate the determinant of A by using the permutation formula.

(c) Calculate the determinant of A by cofactor expansion down the fourth column.

Solution to (c). Using cofactor expansion down the fourth column,

$$\begin{aligned}
\det(A) &= (-1)^{1+4}\cdot 1\cdot \det\begin{pmatrix} 3 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & -4 & 2 \end{pmatrix} + 0 + 0 \\
&\qquad + (-1)^{4+4}\cdot 4\cdot \det\begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix} \\
&= -\Big((-1)^{1+1}\cdot 3\cdot \det\begin{pmatrix} 0 & 1 \\ -4 & 2 \end{pmatrix} + (-1)^{2+1}\cdot 1\cdot \det\begin{pmatrix} 2 & 2 \\ -4 & 2 \end{pmatrix} + 0\Big) \\
&\qquad + 4\Big((-1)^{1+1}\cdot 1\cdot \det\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} + (-1)^{1+2}\cdot(-2)\cdot \det\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} + 0\Big) \\
&= -3(0+4) + (4+8) + 4\big((2-0) + 2(2-3)\big) \\
&= -12 + 12 + 8 + 8 = 16.
\end{aligned}$$

□