2.10 Week 10 Tutorial: Volumes of paralleliipeds

Let $u_1, u_2, u_3, v_1, v_2, v_3, w_1, w_2, w_3 \in \mathbb{R}$. Define

$$\det(u_1) = u_1, \qquad \det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$$

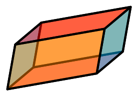
and

$$\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1).$$

Theorem 2.4. ***(Volumes of parallelipipeds.)***

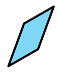
(3) Let $u = |u_1, u_2, u_3\rangle \in \mathbb{R}^3$ *and* $v = |v_1, v_2, v_3\rangle \in \mathbb{R}^3$ *and* $w = |w_1, w_2, w_3\rangle \in \mathbb{R}^3$*. The volume of the parallelipiped with vertices* $0, u, v, w, u+v, u+w, v+w, u+v+w$ *is*

$$\left|\det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\right|.$$



(2) Let $u = |u_1, u_2\rangle \in \mathbb{R}^2$ *and* $v = |v_1, v_2\rangle \in \mathbb{R}^2$*. The area of the paralellogram with vertices* $0, u, v, u+v$ *is*

$$\left|\det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\right|.$$



(1) Let $u = |u_1\rangle \in \mathbb{R}^1$*. The length of the segment with endpoints* 0 *to* u *is*

$$|\det(u_1)|.$$

Proof. Let E_{ij} be the 3×3 matrix with 1 in the (i,j) entry and 0 elsewhere. Let $1 = E_{11} + E_{22} + E_{33}$,

$$\begin{aligned} s_{ij} &= 1 - E_{ii} - E_{jj} + E_{ij} + E_{ji}, && \text{for } i,j \in \{1,\ldots,3\} \text{ with } i \neq j, \\ x_{ij}(c) &= 1 + cE_{ji}, && \text{for } i,j \in \{1,\ldots,3\} \text{ with } i \neq j \text{ and } c \in \mathbb{R}, \\ d_i(c) &= 1 + (c-1)E_{ii}, && \text{for } i \in \{1,\ldots,3\} \text{ and } c \in \mathbb{R} \text{ with } c \neq 0. \end{aligned}$$

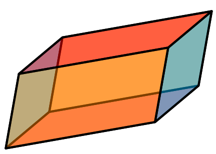
Consider the volume of the parallelipiped as a function of the edge vectors $u = |u_1, u_2, u_3\rangle$, $v = |v_1, v_2, v_3\rangle$ and $w = |w_1, w_2, w_3\rangle$.

$$\mathrm{Vol}(u,v,w) = \mathrm{Vol}\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \mathrm{Vol}(P), \qquad \text{where} \quad P = \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

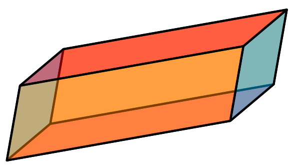
Since switching two edges produces the same paralleliipiped then

$$\mathrm{Vol}(s_{ij}P) = \mathrm{Vol}(P).$$

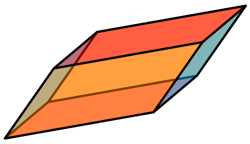
Since stretching one of the edges of the parallelipiped by a factor of c changes the volume by a factor of c then



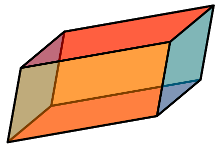
$$\mathrm{Vol}(d_i(c)P) = c \cdot \mathrm{Vol}(P).$$



Since area of a parallelogram is always (area of base) · (height) then two paralellipipeds of the same height and the the same base have the same volume and



$$\mathrm{Vol}(x_{ij}(c)P) = \mathrm{Vol}(P).$$



It follows that $\mathrm{Vol}(P)$ can be computed by writing P as a product of elementary matrices and using

$$\mathrm{Vol}(s_{ij}) = 1, \qquad \mathrm{Vol}(d_i(c)) = c, \qquad \mathrm{Vol}(x_{ij}(c)) = 1.$$

Thus the volume of P is the absolute value of the determinant of P,

$$\mathrm{Vol}(P) = |\det(P)|.$$

$\square$

Example E7. Find the volume of the parallelipiped with adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$, where

$$P = |2, 0, -1\rangle, \qquad Q = |4, 1, 0\rangle, \qquad R = |3, -1, 1\rangle \qquad \text{and} \qquad S = |2, -2, 2\rangle.$$

Since the edges of the parallelipiped are

$$\overrightarrow{PQ} = P - Q = |2, 1, 1\rangle, \qquad \overrightarrow{PR} = P - R = |1, -1, 2\rangle, \qquad \overrightarrow{PS} = P - S = |0, -2, 3\rangle,$$

then

$$\begin{aligned}
(\text{Volume of parallelipiped}) &= |\langle \overrightarrow{PQ}, \overrightarrow{PR} \times \overrightarrow{PS}\rangle| \\
&= \left| \det \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{pmatrix} \right| = \left| 2 \cdot \det \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} - \det \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \right| \\
&= |2(-3+4) - (3+2)| = |-3| = 3.
\end{aligned}$$

$\square$

Example E8. Find the area of the triangle in $\mathbb{R}^3$ with vertices $|2, -5, 4\rangle$, $|3, -4, 5\rangle$ and $|3, -6, 2\rangle$. Letting

$$u = |3, -4, 5\rangle - |2, -5, 4\rangle = |1, 1, 1\rangle \qquad \text{and} \qquad v = |3, -6, 2\rangle - |2, -5, 4\rangle = |1, -1, -2\rangle,$$

then

$$\begin{aligned}
u \times v &= |1, 1, 1\rangle \times |1, -1, -2\rangle \\
&= |1 \cdot (-2) - 1 \cdot (-1), -(1 \cdot (-2) - 1 \cdot 1), 1 \cdot (-1) - 1 \cdot 1\rangle \\
&= |-1, 3, -2\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
(\text{Area of triangle}) &= \tfrac{1}{2}(\text{area of rectangle with edges } u \text{ and } v) \\
&= \tfrac{1}{2} \frac{1}{\|u \times v\|} \begin{pmatrix} \text{volume of parallelipiped} \\ \text{with edges } u,\ v \text{ and } u \times v) \end{pmatrix} \\
&= \tfrac{1}{2} \frac{1}{\|u \times v\|} \langle u \times v, u \times v\rangle \\
&= \tfrac{1}{2} \|u \times v\| = \tfrac{1}{2} \|\, |-1, 3, -2\rangle\, \| \\
&= \tfrac{1}{2} \sqrt{(-1)^2 + 3^2 + (-2)^2} = \frac{\sqrt{14}}{2}. \qquad \square
\end{aligned}$$

2.10.1 Cross products in $\mathbb{R}^3$

Proposition 2.5. *Let $u = |u_1, u_2, u_3\rangle$, $v = |v_1, v_2, v_3\rangle$, $w = |w_1, w_2, w_3\rangle$ in $\mathbb{R}^3$ and let $\theta(v, w)$ be the angle between v and w. Then*

$$\langle u|v \times w\rangle = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}, \qquad \|v \times w\| = \|v\| \cdot \|w\| \sin(\theta(v, w)), \tag{crosslengthB}$$

$$\text{and} \quad \hat{n} = \frac{1}{\|v \times w\|} v \times w \quad \text{is a unit vector orthogonal to } v \text{ and } w.$$

Proof. (a) Then

$$\langle u|v \times w\rangle = u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

(b) Since switching rows changes the sign of the determinant then

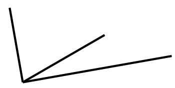
$$\langle u|v \times w\rangle = -\langle v|u \times w\rangle = -\langle w|u \times v\rangle.$$

Since $\langle u|u \times v\rangle = -\langle u|u \times v\rangle$ and $\langle v|u \times v\rangle = -\langle v|u \times v\rangle$ then

$$\langle u|u \times v\rangle = 0 \qquad \text{and} \qquad \langle v|u \times v\rangle = 0,$$

giving that $u \times v$ is perpendicular to u and $u \times v$ is perpendicular to v. Thus

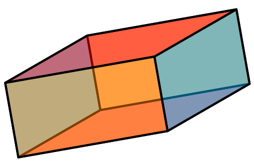
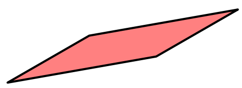
$$\hat{n} = \frac{1}{\|v \times w\|} v \times w \quad \text{is a unit vector orthogonal to } v \text{ and } w.$$



(c) Let

$$\hat{n} = \frac{1}{\|v \times w\|} v \times w = |n_1, n_2, n_3\rangle,$$

so that n_1, n_2 and n_3 are the coordinates of a unit vector orthogonal to v and w. Let R be the parallelogram with vertices $(0, 0, 0)$, (v_1, v_2, v_3), (w_1, w_2, w_3) and $(v_1 + w_1, v_2 + w_2, v_3 + w_3)$. Let P be the parallelipiped with vertices $(0, 0, 0)$, (n_1, n_2, n_3), (v_1, v_2, v_3), (w_1, w_2, w_3), $(n_1 + v_1, n_2 + v_2, n_3 + v_3)$, $(n_1 + w_1, n_2 + w_2, n_3 + w_3)$, $(v_1 + w_1, v_2 + w_2, v_3 + w_3)$ and $(n_1 + v_1 + w_1, n_2 + v_2 + w_2, n_3 + v_3 + w_3)$.

$P =$  and $R =$ 

Using Theorem 2.4,

$$(\text{Area of } R) = 1 \cdot (\text{Area of } R) = (\text{Volume of } P) = \left| \det\begin{pmatrix} n_1 & v_1 & w_1 \\ n_2 & v_2 & w_2 \\ n_3 & v_3 & w_3 \end{pmatrix} \right|$$

$$= \langle \hat{n}|v \times w\rangle = \frac{1}{\|v \times w\|}\langle v \times w|v \times w\rangle = \|v \times w\|.$$

Thus

$$\|v \times w\| = (\text{Area of } R) = (\text{base}) \cdot (\text{height}) = \|v\| \cdot \|w\| \sin(\theta), \tag{crosslength}$$

where θ is the angle between v and w. $\square$