1 Matrix operations

1.1 Addition, scalar multiplication and multiplication

1. Calculate

$$\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \end{pmatrix}.$$

2. Calculate

$$\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \\ -1 & -2 & -3 & -4 \end{pmatrix}.$$

3. Explain the meaning of 0 in the following expression and compute the sum.

$$0 + \begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -2 & i\sqrt{2} \\ -4.1 & -3 \end{pmatrix}$$

4. Explain the meaning of the negative before the first matrix in the following expression and compute the sum.

$$-\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix}$$

5. Calculate

$$2i \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & 3.2 & i\sqrt{2} & 0 \end{pmatrix}.$$

6. Calculate

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2i \end{pmatrix}.$$

7. Calculate

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & 0 & -2i \end{pmatrix}.$$

8. Using that $i^2 = -1$, compute

$$\begin{pmatrix} 4 & 2 & 6 & 3 \\ 1 & 2 & -3 & 2i \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ -2i & 0 & 1 \end{pmatrix}.$$

9. Using $i^2 = -1$, compute

$$\begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 6 & -3 \\ 3 & 2i \end{pmatrix}.$$

10. Explain the meaning of 1 and compute the products

$$1 \cdot \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot 1.$$

11. Compute the following products.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

12. Compute the following products

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{8} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{7} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{7} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{8} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

13. Compute the following products.

$$\begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

14. Calculate

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

15. Compute

$$\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 \end{pmatrix}.$$

16. Compute
$$\frac{1}{3}\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}.$$

17. Let
$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Compute $78E_{11} + 62E_{12} + 91E_{13} + 32E_{21} + 41E_{22} + 24E_{23}$.

18. Compute
$$\begin{pmatrix} 2 & 5 & 11 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 3 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}\begin{pmatrix} \frac{2}{100} \\ \frac{85}{100} \\ \frac{1}{100} \\ \frac{12}{100} \end{pmatrix}.$$

19. Find $A \in M_2(\mathbb{Q})$ and $B \in M_2(\mathbb{Q})$ such that $AB \neq BA$.

20. Calculate the inner product $\langle (4, 2, 6, 3)^t, (-1, 2, 0, -2i)^t \rangle$ using matrix multiplication.

21. Prove that
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 0.$$

22. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 2 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 3 \end{pmatrix},$$
$$D = \begin{pmatrix} -1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} -5 & 2 \\ 4 & 0 \\ 3 & 1 \end{pmatrix}.$$
Calculate $A + A$, $A + B$, $A + C$, $A + D$, $A + E$, $B + B$, $B + C$, $B + D$, $B + E$, $C + C$, $C + D$, $C + E$, $D + D$, $D + E$ and $E + E$.

23. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 2 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 3 \end{pmatrix},$$
$$D = \begin{pmatrix} -1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} -5 & 2 \\ 4 & 0 \\ 3 & 1 \end{pmatrix}.$$
Calculate AA, AB ,AC, AD, AE, BB, BC, BD, BE, CC, CD, CE, DD, DE and EE.

24. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 & 1 \\ 5 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 2 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 3 \end{pmatrix},$$
$$D = \begin{pmatrix} -1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \qquad E = \begin{pmatrix} -5 & 2 \\ 4 & 0 \\ 3 & 1 \end{pmatrix}.$$
Calculate BA ,CA, DA, EA, CB, DB, EB, DC, EC and ED.

25. Let $A = \begin{pmatrix} 7 & 4 \\ -9 & -5 \end{pmatrix}$.

   (a) Show that if $n \in \mathbb{Z}_{>0}$ then $A^n = \begin{pmatrix} 1+6n & 4n \\ -9n & 1-6n \end{pmatrix}$.

   (b) Show that if $n \in \mathbb{Z}_{>0}$ then $A^{-n} = \begin{pmatrix} 1+6(-n) & 4(-n) \\ -9(-n) & 1-6(-n) \end{pmatrix}$.

1.2 Matrix equations

26. Find a matrix $A \in M_2(\mathbb{Q})$ such that A is not diagonal and $A^2 = 1$.

27. Let $a, b, c, d \in \mathbb{C}$. Let
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and assume that} \qquad AB = BA.$$
Show that $b = 0$ and $c = 0$.

28. Let $Z = \{A \in M_2(\mathbb{C}) \mid \text{if } B \in M_2(\mathbb{C}) \text{ then } AB = BA\}$.
$$\text{Show that} \qquad Z \subseteq \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \;\middle|\; a \in \mathbb{C} \right\}.$$

29. Use that $X \cdot X^2 = X^2 \cdot X$ to show that
$$\text{if} \quad X^2 = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \qquad \text{then} \qquad X \in \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{C} \right\}.$$

30. Let
$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$
Determine the entries of the elements of $\{A \in M_n(\mathbb{C}) \mid AB = BA\}$.

31. Let $c_1, c_2 \in \mathbb{Q}$ with $c_2 \neq 0$. Prove that
$$\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_1 + c_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 0 \\ 0 & -c_2^{-1} \end{pmatrix} \begin{pmatrix} 1 & c_2^{-1} \\ 0 & 1 \end{pmatrix}.$$

32. Let $c_1 \in \mathbb{Q}$. Prove that
$$\begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}.$$

33. Let $c_1, c_2, c_3, d_1, d_2, d_3 \in \mathbb{Q}$. Prove that
$$\begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}.$$

34. Let $c_1, c_2, c_3 \in \mathbb{Q}$. Prove that
$$\begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1c_3 + c_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

35. Let
$$s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$
Prove that $s_1s_2s_1 = s_2s_1s_2$.

36. Let
$$s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
Prove that $s_1s_3 = s_3s_1$.

37. For $c \in \mathbb{Q}$ let
$$s_1(c) = \begin{pmatrix} c & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s_3(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
Prove that if $c_1, c_2 \in \mathbb{Q}$ then $s_1(c_1)s_3(c_2) = s_3(c_2)s_1(c_1)$.

38. Let $c, d_1, d_2 \in \mathbb{Q}$ with $d_2 \neq 0$. Prove that
$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} cd_1d_2^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} d_2 & 0 \\ 0 & d_1 \end{pmatrix}.$$

39. Show that
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

40. Let $c \in \mathbb{Q}$. Show that

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

41. Let $n \in \mathbb{Z}_{>0}$. Prove that, in $M_n(\mathbb{Q})$,

$$E_{ij}E_{j\ell} = E_{i\ell} \qquad \text{and} \qquad \text{if } j \neq k \text{ then } E_{ij}E_{k\ell} = 0.$$

42. Let $n \in \mathbb{Z}_{\geq 2}$ and $i, j \in \{1, \ldots, n\}$. Prove that, in $M_n(\mathbb{Q})$, E_{ij} is not invertible.

43. Let $n \in \mathbb{Z}_{>0}$. For $c \in \mathbb{Q}$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ define $x_{ij}(c) \in M_n(\mathbb{Q})$ by

$$x_{ij}(c) = 1 + cE_{ij}.$$

Prove that $x_{ij}(c)^{-1} = x_{ij}(-c)$.

44. Let $n \in \mathbb{Z}_{>0}$. For $c_1, \ldots, c_n \in \mathbb{Q}$ define $d(c_1, \ldots, c_n) \in M_n(\mathbb{Q})$ by

$$d(c_1, \ldots, c_n) = c_1E_{11} + \cdots + c_nE_{nn}.$$

Prove that if $c_1^{-1}, \ldots, c_n^{-1} \in \mathbb{Q}$ then $d(c_1, \ldots, c_n)^{-1} = d(c_1^{-1}, \ldots, c_n^{-1})$.

45. Let $n \in \mathbb{Z}_{>0}$. For $i \in \{1, \ldots, n-1\}$ define $s_i \in M_n(\mathbb{Q})$ by

$$s_i = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i}.$$

Prove that if $i \in \{1, \ldots, n-1\}$ then $s_i^2 = 1$.

46. Let $n \in \mathbb{Z}_{>0}$. For $i \in \{1, \ldots, n-1\}$ define $s_i \in M_n(\mathbb{Q})$ by

$$s_i = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i}.$$

Prove that

$$\text{if } i, j \in \{1, \ldots, n-1\} \text{ and } j \notin \{i-1, i+1\} \quad \text{then} \quad s_is_j = s_js_i.$$

47. Let $n \in \mathbb{Z}_{>0}$. For $i \in \{1, \ldots, n-1\}$ define $s_i \in M_n(\mathbb{Q})$ by

$$s_i = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i}.$$

Prove that

$$\text{if } i \in \{1, \ldots, n-2\} \quad \text{then} \quad s_is_{i+1}s_i = s_{i+1}s_is_{i+1}.$$

48. Let $n \in \mathbb{Z}_{>0}$. For $c \in \mathbb{Q}$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ define $x_{ij}(c), d_i(c) \in M_n(\mathbb{Q})$ by

$$x_{ij}(c) = 1 + cE_{ij} \qquad \text{and} \qquad d_i(c) = 1 + (c-1)E_{ii}.$$

For $i \in \{1, \ldots, n-1\}$ and $c \in \mathbb{Q}$ define

$$s_i = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} \qquad \text{and} \qquad s_i(c) = x_{i,i+1}(c)s_i.$$

Prove that if $c_1, c_2 \in \mathbb{Q}$ then

$$s_i(c_1)s_i(c_2) = \begin{cases} s_i(c_1 + c_2^{-1})d_i(c_2)d_{i+1}(-c_2^{-1})x_{i,i+1}(c_2^{-1}), & \text{if } c_2 \neq 0, \\ x_{i,i+1}(c_1), & \text{if } c_2 = 0. \end{cases}$$

49. Let $n \in \mathbb{Z}_{>0}$. For $c \in \mathbb{Q}$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ define $x_{ij}(c), d_i(c) \in M_n(\mathbb{Q})$ by

$$x_{ij}(c) = 1 + cE_{ij} \qquad \text{and} \qquad d_i(c) = 1 + (c-1)E_{ii}.$$

For $i \in \{1, \ldots, n-1\}$ and $c \in \mathbb{Q}$ define

$$s_i = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} \qquad \text{and} \qquad s_i(c) = x_{i,i+1}(c)s_i.$$

Prove that if $c_1, c_2 \in \mathbb{Q}$ and $i, j \in \{1, \ldots, n-1\}$ with $j \notin \{i-1, i, i+1\}$ then

$$s_i(c_1)s_j(c_2) = s_j(c_2)s_i(c_1).$$

50. Let $n \in \mathbb{Z}_{>0}$. For $c \in \mathbb{Q}$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ define $x_{ij}(c), d_i(c) \in M_n(\mathbb{Q})$ by

$$x_{ij}(c) = 1 + cE_{ij} \qquad \text{and} \qquad d_i(c) = 1 + (c-1)E_{ii}.$$

For $i \in \{1, \ldots, n-1\}$ and $c \in \mathbb{Q}$ define

$$s_i = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} \qquad \text{and} \qquad s_i(c) = x_{i,i+1}(c)s_i.$$

Prove that if $c_1, c_2, c_3 \in \mathbb{Q}$ and $i \in \{1, \ldots, n-2\}$ then

$$s_i(c_1)s_{i+1}(c_2)s_i(c_3) = s_{i+1}(c_3)s_i(c_1c_3 + c_2)s_{i+1}(c_1).$$

51. Let $n \in \mathbb{Z}_{>0}$. For $c, t_1, \ldots, t_n \in \mathbb{Q}$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ define $x_{ij}(c), d_i(c) \in M_n(\mathbb{Q})$ by

$$x_{ij}(c) = 1 + cE_{ij} \qquad \text{and} \qquad d(t_1, \ldots, t_n) = t_1E_{11} + \cdots + t_nE_{nn}.$$

Prove that if $t_j \neq 0$ then

$$d(t_1, \ldots, t_n)x_{ij}(c) = x_{ij}(ct_it_j^{-1})d(t_1, \ldots, t_n).$$

52. Let $n \in \mathbb{Z}_{>0}$. For $c \in \mathbb{Q}$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ define $x_{ij}(c), d_i(c) \in M_n(\mathbb{Q})$ by

$$x_{ij}(c) = 1 + cE_{ij} \qquad \text{and} \qquad d(t_1, \ldots, t_n) = t_1E_{11} + \cdots + t_nE_{nn}.$$

For $i \in \{1, \ldots, n-1\}$ and $c \in \mathbb{Q}$ define

$$s_i = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} \qquad \text{and} \qquad s_i(c) = x_{i,i+1}(c)s_i.$$

Prove that if $t_j \neq 0$ then

$$d(t_1, \ldots, t_n)s_i(c) = s_i(ct_it_j^{-1})d(t_1, \ldots, t_{i-1}, t_{i+1}, t_i, \ldots, t_n).$$

53. For $c \in \mathbb{Q}$ let

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Correct the following statement

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 1 & 0 & 2 \end{pmatrix} = s_3(-1)s_2(2) \begin{pmatrix} -1 & 1 & 0 & 2 \\ 32 & 21 & 16 & 5 \\ 0 & 3 & 0 & 5 \end{pmatrix}.$$

1.3 Matrix operations: Definitions and results

54. Carefully define matrices, $M_{t\times s}(\mathbb{Q})$ and $M_{n\times n}(\mathbb{Q})$.
55. Carefully define addition, scalar multiplication and multiplication for matrices.
56. Carefully define the zero matrix, the identity matrix and the negative of matrix.
57. Carefully define the transpose of a matrix.
58. Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with matrices in $\mathbb{Q}$. Prove that
$$\text{if } A, B, C \in M_{m\times n}(\mathbb{Q}) \quad \text{then} \quad A + (B + C) = (A + B) = C.$$
59. Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$. Prove that
$$\text{if } A, B \in M_{m\times n}(\mathbb{Q}) \text{ then } A + B = B + A.$$
60. Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$. Prove that
$$\text{if } A \in M_{m\times n}(\mathbb{Q}) \text{ then } 0 + A = A \text{ and } A + 0 = A.$$
61. Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m\times n}(\mathbb{Q})$ be the set of $m\times n$ matrices with entries in $\mathbb{Q}$. If $A \in M_{m\times n}(\mathbb{Q})$ then $(-A) + A = 0$ and $A + (-A) = 0$.
62. Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$. Prove that if $A \in M_{m\times n}(\mathbb{Q})$ and $c_1, c_2 \in \mathbb{Q}$ then
$$c_1 \cdot (c_2 \cdot A) = (c_1c_2) \cdot A.$$
63. Let $m, n \in \mathbb{Z}_{>0}$ and let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$. Prove that if $A \in M_{m\times n}(\mathbb{Q})$ and $1 \in \mathbb{Q}$ is the identity in $\mathbb{Q}$ then
$$1 \cdot A = A \qquad \text{and} \qquad A \cdot 1 = A.$$
64. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A, B, C \in M_n(\mathbb{Q}) \quad \text{then} \quad A + (B + C) = (A + B) + C.$$
65. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A, B \in M_n(\mathbb{Q}) \quad \text{then} \quad A + B = B + A.$$
66. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A \in M_n(\mathbb{Q}) \quad \text{then} \quad 0 + A = A \quad \text{and} \quad A + 0 = A.$$
67. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A \in M_n(\mathbb{Q}) \quad \text{then} \quad (-A) + A = 0 \quad \text{and} \quad A + (-A) = 0.$$

68. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A, B, C \in M_n(\mathbb{Q}) \quad \text{then} \quad A(BC) = (AB)C.$$

69. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A, B, C \in M_n(\mathbb{Q}) \quad \text{then} \quad (A+B)C = AC + BC \quad \text{and} \quad C(A+B) = CA + CB.$$

70. Let $n \in \mathbb{Z}_{>0}$ and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A \in M_n(\mathbb{Q}) \quad \text{then} \quad 1A = A \quad \text{and} \quad A1 = A.$$

71. Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{F})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$, and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A, B \in M_{m\times n}(\mathbb{Q}) \quad \text{then} \quad (A+B)^t = A^t + B^t,$$

72. Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$, and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A \in M_{m\times n}(\mathbb{Q}) \text{ and } c \in \mathbb{Q} \quad \text{then} \quad (c\cdot A)^t = c\cdot A^t,$$

73. Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$, and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A, B \in M_n(\mathbb{Q}) \qquad \text{then} \qquad (AB)^t = B^t A^t.$$

74. Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$, and let $M_n(\mathbb{Q})$ be the set of $n \times n$ matrices in $\mathbb{Q}$. Prove that
$$\text{if } A \in M_n(\mathbb{Q}) \quad \text{then} \quad (A^t)^t = A.$$

75. Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$. Prove that
$$M_{m\times n}(\mathbb{Q}) = \left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} E_{ij} \ \Big|\ c_{ij} \in \mathbb{Q} \right\};$$

76. Let $m, n \in \mathbb{Z}_{>0}$, let $M_{m\times n}(\mathbb{Q})$ be the set of $m \times n$ matrices with entries in $\mathbb{Q}$. Prove that if $c_{11}, \ldots, c_{mn} \in \mathbb{Q}$ and
$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} E_{ij} = 0 \qquad \text{then} \qquad \text{if } k \in \{1, \ldots, m\} \text{ and } \ell \in \{1, \ldots, n\} \text{ then } c_{k\ell} = 0.$$

77. (Direct sum of matrices) Let $A_1 \in M_{n_1}(\mathbb{F})$ and let $A_2 \in M_{n_2}(\mathbb{F})$. The *direct sum* of A_1 and A_2 is the matrix $A_1 \oplus A_2 \in M_{n_1+n_2}(\mathbb{F})$ given by
$$(A_1 \oplus A_2)(i,j) = \begin{cases} A_1(i,j), & \text{if } i, j \in \{1, \ldots, n_1\}, \\ A_2(i-n_1, j-n_1), & \text{if } i, j \in \{n_1+1, \ldots, n_1+n_2\}, \\ 0, & \text{otherwise.} \end{cases}$$

In tabular form,

$$A_1 \oplus A_2 = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right).$$

Let $A_1, B_1 \in M_{n_1}(\mathbb{F})$ and let $A_2, B_2 \in M_{n_2}(\mathbb{F})$. Show that

$$(A_1 \oplus A_2) + (B_1 \oplus B_2) = (A_1 + B_1) \oplus (A_2 + B_2) \quad \text{and}$$
$$(A_1 \oplus A_2)(B_1 \oplus B_2) = (A_1B_1) \oplus (A_2B_2).$$

In other words, show that

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right) + \left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right) = \left(\begin{array}{c|c} A_1 + B_1 & 0 \\ \hline 0 & A_2 + B_2 \end{array}\right) \quad \text{and}$$

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)\left(\begin{array}{c|c} B_1 & 0 \\ \hline 0 & B_2 \end{array}\right) = \left(\begin{array}{c|c} A_1B_1 & 0 \\ \hline 0 & A_2B_2 \end{array}\right).$$

78. (Tensor product of matrices) Let $A \in M_{m\times n}(\mathbb{F})$ and $B \in M_{k\times l}(\mathbb{F})$. The *tensor product* of A and B is the matrix $A \otimes B \in M_{mk\times nl}(\mathbb{F})$ given by

$$(A \otimes B)((i,j),(r,s)) = A(i,j)B(r,s),$$

for $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$, $r \in \{1,\ldots,k\}$, $s \in \{1,\ldots,l\}$. In tabular form, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & a_{12} & \cdots & a_{mn} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1\ell} \\ \vdots & & & \vdots \\ b_{k1} & b_{12} & \cdots & b_{k\ell} \end{pmatrix}$$

then

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ \vdots & & & \vdots \\ a_{m1}B & a_{12}B & \cdots & a_{mn}B \end{pmatrix}.$$

Show that if $A, A_1, A_2 \in M_n(\mathbb{F})$ and $B, B_1, B_2 \in M_k(\mathbb{F})$ then

$$(A_1 + A_2) \otimes B = A_1 \otimes B + A_2 \otimes B, \qquad A \otimes (B_1 + B_2) = A \otimes B_1 + A \otimes B_2,$$

and

$$(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1A_2 \otimes B_1B_2).$$

79. (The bracket on matrices) Let $A, B \in M_n(\mathbb{F})$. The *bracket* of A and B is

$$[A,B] = AB - BA.$$

Let $X \in M_n(\mathbb{F})$. Define

$$\mathrm{ad}_X \colon M_n(\mathbb{F}) \to M_n(\mathbb{F}) \qquad \text{by} \qquad \mathrm{ad}_X(Y) = [X,Y].$$

(a) Show that if $X, Y, Z \in M_n(\mathbb{F})$ then

$$\mathrm{ad}_X([Y,Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)].$$

(b) Give an example of $X, Y, Z \in M_n(\mathbb{F})$ such that $[X,[Y,Z]] \neq [[X,Y],Z]$.

(c) Show that if $X, Y, Z \in M_n(\mathbb{F})$ then

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0.$$

1.4 Symmetric, skew-symmetric, Hermitian, orthogonal and unitary

80. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Compute A^T.

81. Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{C})$.
    (a) Define what it means for A to be a symmetric matrix.
    (b) Define what it means for A to be a skew-symmetric matrix.
    (c) Define what it means for A to be a Hermitian matrix.
    (d) Define what it means for A to be an orthogonal matrix.
    (e) Define what it means for A to be a unitary matrix.

82. Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$. Prove that A is Hermitian and B is not Hermitian.

83. Prove that the matrix $U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$ is unitary.

84. Let $\theta \in \mathbb{R}$. Prove that $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

85. Assume $Q \in O_n(\mathbb{R})$. Prove that $\det(Q) \in \{1, -1\}$.

86. Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{Q})$.
    (a) Show that $A^t A$ is a symmetric matrix.
    (b) Show that $A + A^t$ is a symmetric matrix.
    (c) Show that $A - A^t$ is a skew-symmetric matrix.

87. Let $n \in \mathbb{Z}_{>0}$ and $A \in M_n(\mathbb{Q})$. Show that there exists a symmetric matrix B and a skew symmetric matrix C such that $A = B + C$.

88. Let $n \in \mathbb{Z}_{>0}$. Assume that $A \in M_n(\mathbb{Q})$ is a symmetric matrix and $B \in M_n(\mathbb{Q})$ is a skew-symmetric matrix. Assume that $AB = BA$ and $A + B$ is invertible. Prove that
$$(A+B)^{-1}(A-B) \qquad \text{is orthogonal.}$$

89. Let $n \in \mathbb{Z}_{>0}$. Assume that $A \in M_n(\mathbb{Q})$ is a symmetric matrix and $S \in M_n(\mathbb{Q})$ is a skew-symmetric matrix. Assume that $1 + SA$ is invertible. Prove that
$$L = (1 - SA)(1 + SA)^{-1} \qquad \text{satsifies} \qquad L^t A L = A.$$

90. Let $n \in \mathbb{Z}_{>0}$. Assume that $A \in M_n(\mathbb{Q})$ is a symmetric matrix and $L \in M_n(\mathbb{Q})$. Assume that $1 + L$ and A are invertible. Prove that
$$S = (1 + L)^{-1}(1 - L)A^{-1} \qquad \text{is skew-symmetric.}$$

91. For $\alpha \in \mathbb{C}$ let $A(\alpha) = \begin{pmatrix} 1 & \alpha & \alpha^2/2 \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Show that if $\alpha, \beta \in \mathbb{C}$ then $A(\alpha)A(\beta) = A(\alpha+\beta)$.

(b) Let $\alpha \in \mathbb{C}$. Find the inverse of $A(\alpha)$.

(c) Show that $A(3\alpha) - 3A(2\alpha) + 3A(\alpha) = 1$.

(d) Find a cubic polynomial equation satisfied by $A(\alpha)$.

92. Let $A \in M_n(\mathbb{R})$.
Give a formula for the diagonal entries of A^tA in terms of the entries of A.

93. Let $A, B, C \in M_n(\mathbb{R})$.

(a) Show that if A and B are symmetric and C is skew-symmetric and $A^2 + B^2 = C^2$ then $A = 0$, $B = 0$ and $C = 0$.

(b) Assume B is symmetric and C is skew-symmetric and $A^2 + B^2 = C^2$. Can one conclude that $A = 0$, $B = 0$ and $C = 0$?

94. Assume $X \in M_2(\mathbb{Q})$ and $X^tAX = B$, where

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that there exists $\alpha \in \mathbb{Q}$ such that

$$X = \begin{pmatrix} \alpha & \frac{1}{2}\alpha^{-1} \\ \alpha & -\frac{1}{2}\alpha^{-1} \end{pmatrix} \qquad \text{or} \qquad X = \begin{pmatrix} \alpha & \frac{1}{2}\alpha^{-1} \\ -\alpha & -\frac{1}{2}\alpha^{-1} \end{pmatrix}.$$

(b) Find $\{X \in M_2(\mathbb{Q}) \mid X^tAX = B \text{ and } X^tX = 1\}$.

2 Inverses, normal form and rank

2.1 Inverses

1. Carefully define the inverse of a matrix.

2. Let $n \in \mathbb{Z}_{\geq 2}$ and $i, j \in \{1, \ldots, n\}$. Prove that, in $M_n(\mathbb{Q})$, E_{ij} is not invertible.

3. Let $n \in \mathbb{Z}_{>0}$ and let $GL_n(\mathbb{Q})$ be the set of $n \times n$ mattrices with entries in $\mathbb{Q}$ that are invertible. Prove that if $P, Q \in GL_n(\mathbb{Q})$ then $(PQ)^{-1} = Q^{-1}P^{-1}$.

4. Let $n \in \mathbb{Z}_{>0}$ and let $GL_n(\mathbb{Q})$ be the set of $n \times n$ mattrices with entries in $\mathbb{Q}$ that are invertible. Prove that if $r \in \mathbb{Z}_{>0}$ and $A_1, A_2, \ldots, A_r \in GL_n(\mathbb{Q})$ and $B = A_1A_2 \cdots A_r$ then
$$B^{-1} = A_r^{-1} \cdots A_2^{-1}A_1^{-1}.$$

5. Let $c \in \mathbb{Q}$. Find the inverse of the matrix $\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$.

6. Let $c \in \mathbb{Q}$. FInd the inverses of the matrices
$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

7. Let $d \in \mathbb{Q}$ and assume $d \neq 0$. Find the inverses of the matrices
$$h_1(d) = \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ h_2(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ h_3(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

8. Let $c \in \mathbb{Q}$. Find the inverses of the matrices
$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

9. Find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

10. Find the inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

11. Let $A, B \in GL_n(\mathbb{Q})$. Prove that $AB \in GL_n(\mathbb{Q})$ by finding $(AB)^{-1}$.

12. Let A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Q})$. Prove that if $ad - bc = 0$ then A is not invertible.

13. Let A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2\times 2}(\mathbb{Q})$. Prove that if $ad - bc \neq 0$ then
$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

14. Find the inverse of $\begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

15. Find the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}$.

16. Find the inverse of $\begin{pmatrix} 1 & 3 & -1 \\ 3 & -2 & 7 \\ 4 & -1 & 1 \end{pmatrix}$.

17. Find the inverse of $\begin{pmatrix} 3 & -1 & 4 \\ 5 & 1 & -3 \\ 4 & -1 & 1 \end{pmatrix}$.

18. Find the inverse of $\begin{pmatrix} 0 & -1 & 2 & 1 \\ -4 & 3 & -3 & 5 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 \end{pmatrix}$.

19. Find the inverse of $\begin{pmatrix} 1 & 2 & -3 & 1 \\ 2 & 4 & 1 & 3 \\ -1 & 1 & 1 & 0 \\ -2 & 2 & -5 & -1 \end{pmatrix}$.

20. Find the inverse of $\begin{pmatrix} 1 & i & 0 \\ i & -1 & 1+i \\ 1-i & 0 & 2 \end{pmatrix}$, where $i = \sqrt{-1}$.

21. Find the inverse of $\begin{pmatrix} -2 & 3 & 2 \\ 6 & 0 & 3 \\ 4 & 1 & -1 \end{pmatrix}$.

22. Find the inverse of $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 5 & -2 & 1 \end{pmatrix}$.

23. Find the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}$.

24. Let $n \in \mathbb{Z}_{>0}$. For $c \in \mathbb{Q}$ and $i, j \in \{1, \ldots, n\}$ with $i \neq j$ define $x_{ij}(c) \in M_n(\mathbb{Q})$ by
$$x_{ij}(c) = 1 + cE_{ij}.$$
Prove that $x_{ij}(c)^{-1} = x_{ij}(-c)$.

25. Let $n \in \mathbb{Z}_{>0}$. For $c \in \mathbb{Q}$ and $i \in \{1, \ldots, n\}$ define $d_i(c) \in M_n(\mathbb{Q})$ by
$$d_i(c) = 1 + (c-1)E_{ii}.$$
Prove that

(a) If $c \neq 0$ then $d_i(c)^{-1} = d_i(c)^{-1}$.
(b) If $c = 0$ then $d_i(c)^{-1}$ does not exist.

26. For $i \in \{1, \dots, n-1\}$ and $c \in \mathbb{Q}$ define

$$s_i = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} \qquad \text{and} \qquad s_i(c) = x_{i,i+1}(c)s_i.$$

Determine s_i^{-1} and $s_i(c)^{-1}$.

27. An *ordered basis of* $\mathbb{F}^n$ is a sequence $(p_1, \dots, p_n)$ of column vectors in $\mathbb{F}^n$ such that

if $v \in \mathbb{F}^n$ then there exists unique $c_1, \dots, c_n \in \mathbb{F}$ such that $v = c_1p_1 + \cdots + c_np_m$.

Prove that the function

$$\begin{array}{ccc} \{\text{ordered bases } (p_1, \dots, p_n) \text{ of } \mathbb{F}^n\} & \longrightarrow & GL_n(\mathbb{F}) \\ (p_1, \dots, p_n) & \longmapsto & P = \begin{pmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{pmatrix} \end{array} \qquad \text{is a bijection.}$$

2.2 Normal form

The input to the normal form algorithm is a matrix A. The output of the normal form algorithm is a **factorization of** A as a product of row-reducers times a product of diagonal generators times a product of root generators times a matrix 1_r times a product of row-reducers times a product of root generators. Specifically, if $A \in M_{t\times s}(\mathbb{Q})$ then the output of the normal form algorithm is the sequence of values $c_1, c_2, \cdots, a_1, a_2, \dots, e_{t-1,t}, \dots, e_{12}, f_1, f_2, \dots, g_{t-1,t}, \dots, g_{12}$ such that

$$\begin{aligned} A = {} & s_{n-1}(c_1)s_{n-2}(c_2)\cdots d_1(a_1)d_2(a_2)\cdots x_{t-1,t}(e_{t-1,t})\cdots x_{12}(e_{12})\cdot 1_r \\ & \cdot s_1(f_1)s_2(f_2)\cdots x_{t-1,t}(g_{t-1,t})\cdots x_{12}(g_{12}). \end{aligned}$$

This factorization of A determines invertible matrices P and Q and a matrix R which is in reduced row echelon form so that

$$\begin{aligned} A &= s_{n-1}(c_1)s_{n-2}(c_2)\cdots d_1(a_1)d_2(a_2)\cdots x_{t-1,t}(e_{t-1,t})\cdots x_{12}(e_{12}) \\ &\quad \cdot 1_r \cdot s_1(f_1)s_2(f_2)\cdots x_{t-1,t}(g_{t-1,t})\cdots x_{12}(g_{12}) \\ &= P1_rQ = PR. \end{aligned}$$

The number r is called the *rank of* A.

1. Let $E_{ij} \in M_{5\times 5}(\mathbb{Q})$ be the matrix with 1 in the (i,j) entry and all other entries 0. For $c \in \mathbb{Q}$ and $i \in \{1,2,3,4\}$ let

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

By multiplying out the matrices on the left hand side check that

$$s_4(1)s_3(2)s_2(3)s_1(4)\cdot s_4(5)s_3(6)s_2(7)\cdot s_4(8)s_3(9)\cdot s_4(10) = \begin{pmatrix} 4 & 7 & 9 & 10 & 1 \\ 3 & 6 & 8 & 1 & 0 \\ 2 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. Let $E_{ij} \in M_{5\times 5}(\mathbb{Q})$ be the matrix with 1 in the (i,j) entry and all other entries 0. For $c \in \mathbb{Q}$ and $i \in \{1,2,3,4,5\}$ let
$$d_i(c) = 1 - E_{ii} + cE_{ii}.$$
By multiplying out the matrices on the left hand side check that
By multiplying out the matrices on the left hand side check that
$$h_1(15)h_2(-3)h_3(76)h_4(-19)h_5(2) = \begin{pmatrix} 15 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 76 & 0 & 0 \\ 0 & 0 & 0 & -19 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

3. Let $E_{ij} \in M_{5\times 5}(\mathbb{Q})$ be the matrix with 1 in the (i,j) entry and all other entries 0. For $c \in \mathbb{Q}$ and $i,j \in \{1,2,3,4,5\}$ let
$$x_{ij}(c) = 1 + cE_{ij}.$$
(a) Check that
$$\begin{pmatrix} 1 & -10 & 71 & -302 & 186 \\ 0 & 1 & -8 & 34 & -21 \\ 0 & 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 10 & 9 & 7 & 4 \\ 0 & 1 & 8 & 6 & 3 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
(b) By multiplying out the matrices on the left hand side check that
$$\begin{array}{l} x_{45}(1)x_{35}(2)x_{25}(3) \\ \cdot x_{15}(4)x_{34}(5)x_{24}(6) \\ \cdot x_{14}(7)x_{23}(8)x_{13}(9) \\ \cdot x_{12}(10) \end{array} = \begin{pmatrix} 1 & 10 & 9 & 7 & 4 \\ 0 & 1 & 8 & 6 & 3 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$
(c) By multiplying out the matrices on the left hand side check that
$$\begin{array}{l} x_{12}(10)^{-1}x_{13}(9)^{-1}x_{23}(8)^{-1} \\ \cdot x_{14}(7)^{-1}x_{24}(6)^{-1}x_{34}(5)^{-1} \\ \cdot x_{15}(4)^{-1}x_{25}(3)^{-1}x_{35}(2)^{-1} \\ \cdot x_{45}(1)^{-1} \end{array} = \begin{pmatrix} 1 & -10 & 71 & -302 & 186 \\ 0 & 1 & -8 & 34 & -21 \\ 0 & 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4. For $c \in \mathbb{R}$ let
$$s_1(c) = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}.$$
Show that if $a, b \in RR$ and $b \neq 0$ then
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{a}{b} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} b \\ 0 \end{pmatrix} = s_1(\tfrac{a}{b}) \begin{pmatrix} b \\ 0 \end{pmatrix}.$$

5. For $c \in \mathbb{R}$ let

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Check that if $a_{31} \neq 0$ and $a_{22} - \frac{a_{32}a_{21}}{a_{31}} \neq 0$ then

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = s_2\big(\frac{a_{21}}{a_{31}}\big) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ 0 & a_{22} - \frac{a_{32}a_{21}}{a_{31}} & a_{23} - \frac{a_{33}a_{21}}{a_{31}} \end{pmatrix}$$

$$= s_2\big(\frac{a_{21}}{a_{31}}\big) s_1\big(\frac{a_{11}}{a_{31}}\big) \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{12} - \frac{a_{32}a_{11}}{a_{31}} & a_{13} - \frac{a_{33}a_{11}}{a_{31}} \\ 0 & a_{22} - \frac{a_{32}a_{21}}{a_{31}} & a_{23} - \frac{a_{33}a_{21}}{a_{31}} \end{pmatrix}$$

$$= s_2\big(\frac{a_{21}}{a_{31}}\big) s_1\big(\frac{a_{11}}{a_{31}}\big) s_2\Big(\frac{a_{12} - \frac{a_{32}a_{11}}{a_{31}}}{a_{22} - \frac{a_{32}a_{21}}{a_{31}}}\Big) \begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22} - \frac{a_{32}a_{21}}{a_{31}} & a_{23} - \frac{a_{33}a_{21}}{a_{31}} \\ 0 & 0 & a_{13} - \frac{a_{33}a_{11}}{a_{31}} - \frac{\big(a_{23} - \frac{a_{33}a_{21}}{a_{31}}\big)\big(a_{12} - \frac{a_{32}a_{11}}{a_{31}}\big)}{a_{22} - \frac{a_{32}a_{21}}{a_{31}}} \end{pmatrix}.$$

6. For $c \in \mathbb{R}$ let

$$s_1(c) = \begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}, \qquad h_1(c) = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}, \qquad h_2(c) = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, \qquad x_{12}(c) = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

(a) Check that

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) By multiplying out the matrices on the left hand side check that

$$s_1(\tfrac{1}{3}) h_1(3) h_2(\tfrac{2}{3}) x_{12}(\tfrac{4}{3}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(c) By multiplying out the matrices on the left hand side check that

$$x_{12}(\tfrac{4}{3})^{-1} h_2(\tfrac{2}{3})^{-1} h_1(3)^{-1} s_1(\tfrac{1}{3})^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

7. For $c \in \mathbb{R}$ let

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

and let

$$h_1(c) = \begin{pmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h_3(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$$

and

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Check that

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) By multiplying out the matrices on the left hand side check that

$$s_1(-1)s_2(1)h_1(-1)h_3(-1)x_{23}(3)x_{13}(-1)x_{12}(1) = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

(c) By multiplying out the matrices on the left hand side check that

$$\begin{matrix} x_{12}(1)^{-1}x_{13}(-1)^{-1}x_{23}(3)^{-1} \\ \cdot h_3(-1)^{-1}h_1(-1)^{-1}s_2(1)^{-1}s_1(-1)^{-1} \end{matrix} = \begin{pmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}.$$

8. For $c \in \mathbb{Q}$ let

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

and let

$$h_1(c) = \begin{pmatrix} c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h_3(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{pmatrix}$$

and

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{pmatrix} \in M_{3\times 4}(\mathbb{Q})$.

By multiplying out the matrices on the left hand side check that

$$s_2(0)s_1(1)s_2(2)x_{12}(-3)\cdot 1_2 \cdot x_{23}(1)x_{14}(5)x_{24}(-2) = A.$$

Determine $\mathrm{rank}(A)$.

9. Determine the output of the normal form algorithm if the input is

$$A = \begin{pmatrix} 1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{49}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In other words, find $P, Q \in GL_4(\mathbb{Q})$ and $r \in \{0,1,2,3,4\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

10. Determine the output of the normal form algorithm if the input is

$$A = \begin{pmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_5(\mathbb{Q})$ and $r \in \{0,1,2,3\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

11. Determine the output of the normal form algorithm if the input is

$$A = \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \\ 0 & 0 & 0 & -\frac{71}{40} \end{pmatrix}$$

In other words, find $P \in GL_4(\mathbb{Q})$ and $Q \in GL_4(\mathbb{Q})$ and $r \in \{0,1,2,3,4\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

12. Let $a \in \mathbb{Q}$. Determine the output of the normal form algorithm if the input is

$$A = \begin{pmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 3 & -6 & a \end{pmatrix}$$

In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0,1,2,3\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

13. Let

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

14. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}.$$

Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

15. Let

$$A = \begin{pmatrix} -3 & 1 \\ 4 & -1 \end{pmatrix}.$$

Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

16. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

17. Let
$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 3 & 6 & 4 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_4(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

18. Let
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

19. Let
$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

20. Let
$$A = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

21. Let
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

22. Let $\theta \in \mathbb{R}$ and let
$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_3(\mathbb{R})$ and $Q \in GL_2(\mathbb{R})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

23. Let $c \in RR$ and let
$$A = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{R})$ and $Q \in GL_2(\mathbb{R})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

24. Let
$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

25. Let $c \in \mathbb{R}$ and let
$$A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{R})$ and $Q \in GL_5(\mathbb{R})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

26. Let
$$A = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

27. Let
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

28. Let
$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

29. Let
$$A = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & 2 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

30. Let
$$A = \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

31. Let
$$A = \begin{pmatrix} 1 & 5 \\ 5 & -1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

32. Let
$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

33. Determine the output of the normal form algorithm for the inputs

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

34. et $t \in \mathbb{Q}$ and let

$$A - t = \begin{pmatrix} 1-t & 4 \\ 1 & 1-t \end{pmatrix}.$$

Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

35. Let $t \in \mathbb{Q}$ and let

$$A - t = \begin{pmatrix} 0-t & 1 \\ -1 & 0-t \end{pmatrix}.$$

Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

36. Let $t \in \mathbb{Q}$ and let

$$A - t = \begin{pmatrix} 1-t & 2 \\ 0 & 1-t \end{pmatrix}.$$

Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

37. Let $t \in \mathbb{Q}$ and let

$$A - t = \begin{pmatrix} 1-t & -1 \\ -1 & 1-t \end{pmatrix}.$$

Find the normal form of A. In other words, find $P \in GL_2(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

38. Let $t \in \mathbb{C}$ and let

$$A - t = \begin{pmatrix} 1-t & i \\ -i & 1-t \end{pmatrix}.$$

Find the normal form of A. In other words, find $P \in GL_2(\mathbb{C})$ and $Q \in GL_2(\mathbb{C})$ and $r \in \{0, 1, 2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

39. Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

Find the normal form of A. In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0, 1, 2, 3\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

40. Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 1 & -3 & 0 \end{pmatrix}$$

Find the normal form of A. In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0, 1, 2, 3\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

41. Let
$$A = \begin{pmatrix} 2 & 6 & 9 \\ 0 & 3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$$
Find the normal form of A. In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0,1,2,3\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

42. Let
$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$
Find the normal form of A. In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0,1,2,3\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

43. Let
$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 3 & 2 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0,1,2,3\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

44. Let
$$A = \begin{pmatrix} 2 & 2 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

45. Let
$$A = \begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

46. Let
$$A = \begin{pmatrix} 2 & -6 \\ -1 & 3 \\ 1 & -3 \end{pmatrix}.$$
Find the normal form of A. In other words, find $P \in GL_3(\mathbb{Q})$ and $Q \in GL_2(\mathbb{Q})$ and $r \in \{0,1,2\}$ such that $A = P1_rQ$ and $R = 1_rQ$ is in reduced row echelon form.

47. Let
$$A = \begin{pmatrix} 2 & 4 \\ -1 & 0 \\ 1 & 2 \end{pmatrix}.$$
Find the normal form of A.

48. Let
$$A = \begin{pmatrix} 2 & 6 & 2 \\ 0 & 1 & -1 \\ 0 & 7 & 2 \end{pmatrix}.$$
Find the normal form of A.

49. Let
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \end{pmatrix}.$$
Find the normal form of A.

50. Let
$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \end{pmatrix}.$$
Find the normal form of A.

51. Let $t \in \mathbb{Q}$ and let
$$A - t = \begin{pmatrix} 2-t & -3 & 6 \\ 0 & 5-t & -6 \\ 0 & 1 & 0-t \end{pmatrix}.$$
Find the normal form of A.

52. Let
$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$
Find the normal form of A.

53. Determine the output of the normal form algorithm if the input is
$$A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix}$$

54. Determine the output of the normal form algorithm if the input is
$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

55. Determine the output of the normal form algorithm if the input is
$$A = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

56. Let
$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$
Find the normal form of A.

57. Determine the output of the normal form algorithm if the input is
$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

58. Let
$$A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 4 \end{pmatrix}$$
Find the normal form of A.

59. Let
$$A = \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{pmatrix}.$$
Find the normal form of A.

60. Let
$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Find the normal form of A.

61. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}.$$
Find the normal form of A.

62. Let
$$A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 1 & 0 & 1 & 0 \\ 5 & -3 & 7 & -6 \\ 1 & 1 & -1 & 3 \end{pmatrix}.$$
Find the normal form of A.

63. Let
$$A = \begin{pmatrix} 1 & -1 & 2 & -2 \\ 1 & 0 & 1 & 0 \\ 5 & -3 & 7 & -6 \\ 1 & 1 & -1 & 3 \end{pmatrix}^t.$$
Find the normal form of A.

64. Determine the output of the normal form algorithm if the input is
$$A = \begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

65. Determine the output of the normal form algorithm if the input is
$$A = \begin{pmatrix} 1 & -0.1 & 0.01 & -0.001 \\ 1 & 0 & 0 & 0 \\ 1 & 0.1 & 0.01 & 0.001 \\ 1 & 0.2 & 0.04 & 0.008 \end{pmatrix}$$

66. Determine the output of the normal form algorithm if the input is
$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

67. Determine the output of the normal form algorithm if the input is
$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

68. Use the factoring algorithm to prove the following theorem.

Theorem 2.1. *Let $s, t \in \mathbb{Z}_{>0}$ and let E_{ij} denote the $t \times s$ matrix with 1 in the (i,j) entry and 0 elsewhere.*

Let $A \in M_{t\times s}(\mathbb{F})$. Then there exist
$$r \in \{1, \ldots, \min(s,t)\}, \qquad P \in GL_t(\mathbb{F}), \qquad Q \in GL_s(\mathbb{F})$$
$$\text{such that} \qquad A = P1_rQ, \qquad \text{where} \quad 1_r = E_{11} + \cdots + E_{rr}.$$

69. Prove the following theorem.

Theorem 2.2. *Let $t, s \in \mathbb{Z}_{>0}$. For $r \in \{1, \ldots, \min(t,s)\}$ let $1_r \in M_{t\times s}(\mathbb{F})$ be given by*
$$1_r = E_{11} + \cdots + E_{rr},$$
where E_{ij} has 1 in the (i,j) entry and 0 elsewhere. Let $1_0 = 0$ in $M_{t\times s}(\mathbb{F})$. Then
$$M_{t\times s}(\mathbb{F}) = \bigsqcup_{r=0}^{\min(t,s)} GL_t(\mathbb{F})1_rGL_s(\mathbb{F}),$$
where $GL_t(\mathbb{F})1_rGL_s(\mathbb{F}) = \{P1_rQ \in M_{t\times s}(\mathbb{F}) \mid P \in GL_t(\mathbb{F}), Q \in GL_s(\mathbb{F})\}$.

2.3 Reduced row echelon form

70. Carefully define what is means for a matrix $R \in M_{t\times s}(\mathbb{Q})$ to be in reduced row echelon form.

71. Use the first part of the normal form algorithm to prove the following theorem.

    Theorem 2.3. *Let $t, s \in \mathbb{Z}_{>0}$. Let $\mathcal{E}$ be the set of matrices in $M_{t\times s}(\mathbb{F})$ that are in reduced row echelon form. Then*

    $$M_{t\times s}(\mathbb{F}) = \bigsqcup_{R\in\mathcal{E}} GL_t(\mathbb{F}) \cdot R, \qquad \textit{where} \qquad GL_t(\mathbb{F})R = \{PR \in M_{t\times s}(\mathbb{F}) \mid P \in GL_t(\mathbb{F})\}.$$

72. Row reduce A to reduced row echelon form, where

    $$A = \begin{pmatrix} 0 & 0 & 5 & 35 & -24 & 1 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 3 & 1 & 0 & 1 \end{pmatrix}.$$

    In other words, find the matrix $R = 1_r Q$ where $P \in GL_5(\mathbb{Q})$ and $Q \in GL_6(\mathbb{Q})$ and $r \in \{0, 1, 2, 3, 4, 5\}$ and R is in reduced row echelon form.

73. Find the reduced echelon matrix of $\begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & -1 & 1 \\ 1 & -2 & 1 & 1 \end{pmatrix}$. In other words, find the matrix $R = 1_r Q$ where $P \in GL_3(\mathbb{Q})$ and $Q \in GL_4(\mathbb{Q})$ and $r \in \{0, 1, 2, 3\}$ and R is in reduced row echelon form.

74. Find the reduced echelon matrix of $\begin{pmatrix} 2 & 2 & 5 & 3 \\ 6 & 1 & 5 & 4 \\ 4 & -1 & 0 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$. In other words, find the matrix $R = 1_r Q$ where $P \in GL_4(\mathbb{Q})$ and $Q \in GL_4(\mathbb{Q})$ and $r \in \{0, 1, 2, 3, 4\}$ and R is in reduced row echelon form.

75. Find the reduced echelon matrix of $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 0 & 2 \\ 2 & 1 & -1 \end{pmatrix}$. In other words, find the matrix $R = 1_r Q$ where $P \in GL_4(\mathbb{Q})$ and $Q \in GL_3(\mathbb{Q})$ and $r \in \{0, 1, 2, 3\}$ and R is in reduced row echelon form.

76. Find the reduced echelon matrix of $\begin{pmatrix} 0 & 1 & 1 & 1 & 2 & 2 \\ -1 & 4 & 3 & 3 & 4 & 7 \\ 2 & 1 & 3 & 2 & 8 & 3 \\ 3 & 1 & 4 & -1 & 4 & 0 \\ 5 & 2 & 7 & 0 & 10 & 2 \end{pmatrix}$. In other words, find the matrix $R = 1_r Q$ where $P \in GL_5(\mathbb{Q})$ and $Q \in GL_6(\mathbb{Q})$ and $r \in \{0, 1, 2, 3, 4, 5\}$ and R is in reduced row echelon form.

77. Find the reduced echelon matrix of $\begin{pmatrix} i & 1-i & i & 0 \\ 1 & -2 & 0 & i \\ 1-i & -1+i & 1 & 1 \end{pmatrix}$. In other words, find the matrix $R = 1_r Q$ where $P \in GL_3(\mathbb{C})$ and $Q \in GL_4(\mathbb{C})$ and $r \in \{0, 1, 2, 3, 4, 5\}$ and R is in reduced row echelon form.

78. Find the reduced echelon matrix of $\begin{pmatrix} 1 & 1-\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{2} & -3 & 1+\sqrt{2} & -1-2\sqrt{2} \\ -1 & \sqrt{2} & -1 & 1 \\ \sqrt{2}-2 & -2+4\sqrt{2} & -2-\sqrt{2} & 3+\sqrt{2} \end{pmatrix}$. In other words, find the matrix $R = 1_r Q$ where $P \in GL_4(\mathbb{R})$ and $Q \in GL_4(\mathbb{R})$ and $r \in \{0,1,2,3,4\}$ and R is in reduced row echelon form.

79. A matrix A is *row equivalent* to a matrix B if there exist invertible matrices P such that $B = PA$. Is $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$ row equivalent to $\begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}$? Be sure to justify your answer.

80. A matrix A is *row equivalent* to a matrix B if there exist invertible matrices P such that $B = PA$. Is $\begin{pmatrix} 1 & -1 & 1 & 2 \\ -2 & 3 & 0 & 1 \\ 1 & 0 & -1 & 3 \end{pmatrix}$ row equivalent to $\begin{pmatrix} 0 & -1 & 2 & 3 \\ 1 & 2 & -1 & 0 \\ -2 & -5 & 4 & 3 \end{pmatrix}$? Be sure to justify your answer.

2.4 Rank

81. Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 3 & 1 \end{pmatrix}$. Find the rank of A) In other words, find $r \in \mathbb{Z}_{>0}$ such that there exist invertible matrices P and Q such that $A = P1_rQ$. Justify your answer.

82. Let $A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$. Find the rank of A) In other words, find $r \in \mathbb{Z}_{>0}$ such that there exist invertible matrices P and Q such that $A = P1_rQ$. Justify your answer.

83. Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 2 \\ -1 & 3 & 1 \\ -1 & 0 & 1 \end{pmatrix}$. Find the rank of A) In other words, find $r \in \mathbb{Z}_{>0}$ such that there exist invertible matrices P and Q such that $A = P1_rQ$. Justify your answer.

84. Let $A = \begin{pmatrix} 3 & -1 & 0 & 1 & 1 \\ -2 & 0 & 2 & 1 & -1 \\ 1 & 2 & 0 & 0 & 1 \\ -1 & 1 & -2 & -2 & 0 \end{pmatrix}$. Find the rank of A) In other words, find $r \in \mathbb{Z}_{>0}$ such that there exist invertible matrices P and Q such that $A = P1_rQ$. Justify your answer.

85. A matrix A is *left-right-equivalent* to a matrix B if there exist invertible matrices P and Q such that $B = PAQ$. Determine which of the following matrices are left-right-equivalent:

$$C = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & 2 \\ 0 & 2 & 0 & -4 \end{pmatrix} \qquad E = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -1 & 1 & 1 & -2 \\ 1 & 1 & 3 & -4 \end{pmatrix}$$

$$F = \begin{pmatrix} -1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 2 & -1 \end{pmatrix} \qquad G = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

86. Show that the rank of the matrix $\begin{pmatrix} 1-\sqrt{6} & \sqrt{3} & \sqrt{2} \\ 2 & \sqrt{6} & -\sqrt{2} \\ 1 & -\sqrt{3} & \sqrt{2}-\sqrt{6} \end{pmatrix}$ is 2. Justify your answer.

87. Let $a \in \mathbb{Q}$. Find the rank of $\begin{pmatrix} 1 & 3 & 5 & 6+a \\ 2 & 3 & 4-a & 2 \\ 1 & 1-a & -2 & -5 \\ 1 & 6 & 12 & 19 \end{pmatrix}$. Justify your answer.

88. Let $a \in \mathbb{Q}$. Find the rank of $\begin{pmatrix} 1 & 1-2a & 1 & 1-2a \\ 1+2a & -1-a & -1+2a & 1-a \\ 2-a & 3a & 2a & 2-a \\ 5a & 1-a & 1+3a & 0 \end{pmatrix}$. Justify your answer.

89. Let $a, b \in \mathbb{Q}$ and let $A = \begin{pmatrix} a & b & b & a \\ b & a & -1 & -b \\ a+b & a+b & 2a & -2a \\ -2a & 2a & a+b & a+b \end{pmatrix}$.

(a) Show that if $a+b=0$ or $b=3a$ then the rank of A is 4.
(b) Assume $a+b=0$. Find the rank of A.
(c) Assume $b=3a$. Find the rank of A.

Justify your answers.

90. Let $A = \begin{pmatrix} (1+t)t & t-1 & -t \\ 0 & 2 & -1 \\ -2t & 4-2t & t-2 \end{pmatrix}$.

(a) Determine the values $t \in \mathbb{Q}$ such that the rank of A is less than 3.
(b) For each value $t \in \mathbb{Q}$ such that the rank of A is less than 3 determine the rank and express one of the columns as a linear combination of the others.

Justify your answers.

91. Show that the rank of the matrix $\begin{pmatrix} 1-\sqrt{6} & \sqrt{3} & \sqrt{2} \\ 2 & \sqrt{6} & -\sqrt{2} \\ 1 & -\sqrt{3} & \sqrt{2}-\sqrt{6} \end{pmatrix}$ is 2.

92. Let $a \in \mathbb{Q}$. Find the rank of $\begin{pmatrix} 1 & 3 & 5 & 6+a \\ 2 & 3 & 4-a & 2 \\ 1 & 1-a & -2 & -5 \\ 1 & 6 & 12 & 19 \end{pmatrix}$.

93. Let $a \in \mathbb{Q}$. Find the rank of $\begin{pmatrix} 1 & 1-2a & 1 & 1-2a \\ 1+2a & -1-a & -1+2a & 1-a \\ 2-a & 3a & 2a & 2-a \\ 5a & 1-a & 1+3a & 0 \end{pmatrix}$.

94. Let $a, b \in \mathbb{Q}$ and let $A = \begin{pmatrix} a & b & b & a \\ b & a & -1 & -b \\ a+b & a+b & 2a & -2a \\ -2a & 2a & a+b & a+b \end{pmatrix}$.

(a) Show that if $a+b=0$ or $b=3a$ then the rank of A is 4.

(b) Assume $a+b=0$. Find the rank of A.

(c) Assume $b=3a$. Find the rank of A.

95. Let $A = \begin{pmatrix} (1+t)t & t-1 & -t \\ 0 & 2 & -1 \\ -2t & 4-2t & t-2 \end{pmatrix}$.

(a) Determine the values $t \in \mathbb{Q}$ such that the rank of A is less than 3.

(b) For each value $t \in \mathbb{Q}$ such that the rank of A is less than 3 determine the rank and express one of the columns as a linear combination of the others.

3 Kernels, images and solving linear equations

3.1 Solving systems of linear equations

1. (The number of solutions of a linear system)

   (a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

   (b) Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

   (c) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

2. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x \\ y \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

3. Solve the following system of linear equations.

$$\begin{aligned} 4x - 2y + 5z &= 31, \\ 2x - 3y - 2z &= 13, \\ x - 3y + 2z &= 11. \end{aligned}$$

4. Let $A \in GL_n(\mathbb{Q})$ and let $x, b \in \mathbb{Q}^n$. Compute $\mathrm{Sol}(Ax = b)$.

5. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$.

   (a) Compute $\mathrm{Sol}(Ax = b)$.

   (b) Compute $\mathrm{Sol}(Ax = 0)$.

6. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ 15 \end{pmatrix}$.

   (a) Compute $\mathrm{Sol}(Ax = b)$.

   (b) Compute $\mathrm{Sol}(Ax = 0)$.

7. Let $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

   (a) Compute $\mathrm{Sol}(Ax = b)$.

   (b) Compute $\mathrm{Sol}(Ax = 0)$.

8. Find the values of $a, b \in \mathbb{Q}$ for which the system

$$\begin{aligned} x - 2y + z &= 4, \\ 2x - 3y + z &= 7, \\ 3x - 6y + az &= b, \end{aligned} \qquad \text{has} \qquad \begin{array}{l} \text{(a) no solution,} \\ \text{(b) a unique solution,} \\ \text{(c) LOTS of solutions.} \end{array}$$

9. Find all solutions of the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1, \\ 2x_1 - x_2 + x_3 &= 2, \\ 4x_1 + x_2 - x_3 &= 1. \end{aligned}$$

10. Find all solutions of the following system of linear equations.

$$\begin{aligned} 5x_2 + 35x_3 - 24x_4 &= 1, \\ 2x_1 + x_2 - x_3 + x_4 &= 0, \\ 3x_1 + 2x_2 + 2x_3 - x_4 &= 1, \\ 5x_1 + 3x_2 + x_3 &= 1. \end{aligned}$$

11. Find all solutions of the following system of linear equations.

$$\begin{aligned} x_1 - x_2 + x_3 &= 0, \\ x_1 + x_2 + 2x_3 &= 0, \\ x_1 + 2x_2 - x_3 &= 0, \end{aligned}$$

12. Let $k \in \mathbb{Q}$. Find all solutions of the following system of linear equations.

$$\begin{aligned} 2x_1 + x_2 &= 5, \\ x_1 - 3x_2 &= -1, \\ 3x_1 + 4x_2 &= k, \end{aligned}$$

13. Find all solutions of the system of linear equations $Ax = 0$, where

$$A = \begin{pmatrix} 2 & -1 & 1 & 0 & 1 \\ 1 & 2 & -1 & 2 & 1 \\ 1 & -8 & 5 & -6 & -1 \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

14. Find all solutions of the system of linear equations $Ax = 0$, where

$$A = \begin{pmatrix} 1 & -1 & 0 & 2 & 3 \\ 5 & 2 & 1 & 0 & 1 \\ -1 & 1 & 2 & -3 & 4 \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

15. Find all solutions of the system of linear equations $Ax = 0$, where

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & -1 & 1 \\ -1 & 2 & 0 & 1 \\ 1 & 0 & 2 & -3 \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

16. Find all solutions of the system of linear equations $Ax = 0$, where

$$A = \begin{pmatrix} 5 & -3 & 2 & 1 & 1 & 1 \\ 1 & 0 & -1 & 2 & 1 & -1 \\ 2 & -3 & 5 & -5 & -4 & 6 \\ 3 & -6 & 11 & -12 & -5 & 9 \\ 7 & -3 & 0 & 5 & 3 & -1 \end{pmatrix} \qquad \text{and} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}.$$

17. Find all solutions of the system of linear equations $Ax = b$, where

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ -1 & 1 & 2 & 1 \\ 1 & 0 & 2 & 2 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} -1 \\ 5 \\ 4 \end{pmatrix}.$$

18. Find all solutions of the system of linear equations $Ax = b$, where

$$A = \begin{pmatrix} 2 & 1 & -1 & 1 & 1 \\ 5 & 4 & 2 & 4 & 4 \\ 0 & 1 & 1 & 0 & 2 \\ -1 & 1 & 5 & 1 & 1 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 2 \\ 8 \\ 3 \\ 2 \end{pmatrix}.$$

19. Find all solutions of the system of linear equations $Ax = b$, where

$$A = \begin{pmatrix} 2 & 0 & 1 & -1 & 1 \\ -1 & 0 & 2 & 3 & 4 \\ 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 5 & 4 & 10 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} 4 \\ 1 \\ 2 \\ 7 \end{pmatrix}.$$

20. Let $\lambda, \mu \in \mathbb{Q}$. Find all solutions of the following system of linear equations.

$$\begin{aligned} 2x + 3y + z &= 5, \\ 3x - y + \lambda z &= 2, \\ x + 7y - 6z &= \mu. \end{aligned}$$

21. Let $\lambda, \mu \in \mathbb{Q}$. Find all solutions of the following system of linear equations.

$$\begin{aligned} x + y - 4z &= 0, \\ 2x + 3y + z &= 1, \\ 4x + 7y + \lambda z &= \mu. \end{aligned}$$

22. Let $\lambda, a, b, c \in \mathbb{C}$. Find all solutions of the following system of linear equations.

$$\begin{aligned} x + y + z &= a, \\ \lambda x + 2y + z &= b, \\ \lambda^2 x + 4y + z &= c. \end{aligned}$$

23. Let $\lambda \in \mathbb{Q}$. Find all solutions of the following system of linear equations.

$$\begin{aligned} 5x + 2y - z &= 1, \\ 2x + 3y + 4z &= 7, \\ 4x - 5y + \lambda z &= \lambda - 5. \end{aligned}$$

24. Let $\lambda \in \mathbb{Q}$. Find all solutions of the following system of linear equations.

$$\begin{aligned} x + 2y + \lambda z &= 0, \\ 2x + 3y - 2z &= \lambda, \\ \lambda x + y + \lambda^2 z &= 3. \end{aligned}$$

25. Let $\lambda \in \mathbb{Q}$. Find all solutions of the following system of linear equations.

$$\begin{aligned} x + 5y + 3 &= 0, \\ 5x + y - \lambda &= 0, \\ x + 2y + \lambda &= 0. \end{aligned}$$

26. Let $\lambda, a \in \mathbb{Q}$. Find all solutions of the following system of linear equations.

$$\begin{aligned} 2x + 3y + z + t &= 0, \\ x + 2y + z - t &= 1, \\ 3x + 5y + 2z + t &= a, \\ 6x + 10y + 4z + t &= (\lambda + 1)^2. \end{aligned}$$

27. Find all solutions of the following system of linear equations.

$$\begin{aligned} y + z + u + 2v &= 2, \\ -x + 4y + 3z + 3u + 4v &= 7, \\ 2x + y + 3z + 2u + 8v &= 3, \\ 3x + y + 4z - u + 4v &= 0, \\ 5x + 2y + 7z \quad + 10v &= 2. \end{aligned}$$

28. Let $n = 8$. Solve the system

$$\begin{array}{llllllll} x_1+ & x_2 & & & & & & = 0, \\ x_1+ & x_2+ & x_3 & & & & & = 0, \\ & x_2+ & x_3+ & x_4 & & & & = 0, \\ & & & \vdots & & & & \\ & & & & x_{n-3}+ & x_{n-2}+ & x_{n-1} & = 0, \\ & & & & & x_{n-2}+ & x_{n-1} + x_n & = 0, \\ & & & & & & x_{n-1} + x_n & = 0. \end{array}$$

29. Let $n = 9$. Solve the system

$$\begin{array}{lllllll}
x_1+ & x_2 & & & & & = 0, \\
x_1+ & x_2+ & x_3 & & & & = 0, \\
& x_2+ & x_3+ & x_4 & & & = 0, \\
& & & \vdots & & & \\
& & & x_{n-3}+ & x_{n-2}+ & x_{n-1} & = 0, \\
& & & & x_{n-2}+ & x_{n-1}+x_n & = 0, \\
& & & & & x_{n-1}+x_n & = 0.
\end{array}$$

30. Explain how every system of linear equations can be written as a matrix equation.

31. Carfeully define $\mathrm{Sol}(Ax = b)$.

32. Let $A \in M_{t\times s}(\mathbb{F})$ and $b \in \mathbb{F}^t$. Prove that if $t = s$ and $A \in GL_t(\mathbb{F})$ then $\mathrm{Sol}(Ax = b) = \{A^{-1}b\}$.

33. Let $A \in M_{t\times s}(\mathbb{F})$ and $b \in \mathbb{F}^t$ and assume $\mathrm{Sol}(Ax = b) \neq \emptyset$. Let $p \in \mathrm{Sol}(Ax = b)$. Prove that

$$\mathrm{Sol}(Ax = b) = p + \ker(A).$$

34. Prove the following Proposition.

Proposition 3.1. *Let $A \in M_{t\times s}(\mathbb{F})$ and $b \in \mathbb{F}^t$. Assume $r \in \{1,\ldots,\min(s,t)\}$ and $P \in GL_t(\mathbb{F})$ and $Q \in GL_s(\mathbb{F})$ are such that*

$$A = P1_rQ.$$

Let $q_1,\ldots,q_s$ be the columns of Q^{-1}. Then

$$\mathrm{Sol}(Ax=b) = \begin{cases} Q^{-1}\begin{pmatrix} (P^{-1}b)_1 \\ \vdots \\ (P^{-1}b)_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \mathrm{span}\left\{ \begin{pmatrix} | \\ q_{r+1} \\ | \end{pmatrix}, \ldots, \begin{pmatrix} | \\ q_s \\ | \end{pmatrix} \right\}, & \begin{array}{l}\textit{if the last } t-r \textit{ entries of } P^{-1}b \\ \textit{are equal to } 0,\end{array} \\ \\ \emptyset, & \textit{otherwise.} \end{cases}$$

35. Determine whether the following systems of homogeneous equations has a nontrivial solution.

$$\text{(a)} \quad \begin{aligned} x+2y-z&=0, \\ -2x+y+2z&=0, \\ x-y+z&=0; \end{aligned} \qquad\qquad \text{(b)} \quad \begin{aligned} x+2y-z&=0, \\ -2x+y+2z&=0, \\ x-y-z&=0. \end{aligned}$$

36. (a) Show that if $k, p \in \mathbb{R}$ and $k \neq 1$ then there is always a unique solution to the equations

$$\begin{aligned} 2x-y-z&=6, \\ x-2y+z&=p, \\ x+ky-2z&=p^2. \end{aligned}$$

(b) Find the solution when $k = 2$ and $p = 1$.

(c) Show that if $k = 1$ then there are exactly two values of p for which the equations have solutions. Find this values and solve the equations completely in each case.

37. For which values of k does the following system of linear equations have (i) no solution, (ii) a unique solution and (iii) more than one solution?

$$\begin{aligned} x + (k+2)y + 2z &= 2, \\ (k^2+1)x + 2(k+2)y + 4z &= 4k, \\ 3x + 9y + 3(k+1)z &= 6k. \end{aligned}$$

3.2 Determining kernels and images

38. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$.

    (a) Compute $\ker(A)$.

    (b) Compute $\mathrm{im}(A)$.

    (c) Find $\mathrm{Sol}(Ax = b)$.

39. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ 15 \end{pmatrix}$.

    (a) Compute $\ker(A)$.

    (b) Compute $\mathrm{im}(A)$.

    (c) Find $\mathrm{Sol}(Ax = b)$.

40. Let $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

    (a) Compute $\ker(A)$.

    (b) Compute $\mathrm{im}(A)$.

    (c) Find $\mathrm{Sol}(Ax = b)$.

41. Let $S = \{ |1, 3, -1, 1\rangle,\ |2, 6, 0, 4\rangle,\ |3, 9, -2, 4\rangle\ \}$. Show that

$$\mathbb{R}\text{-span}(S) = \mathrm{im}(A), \text{ where } A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}.$$

42. Let $A \in M_{t \times s}(\mathbb{Q})$ and $P \in GL_t(\mathbb{Q})$ and $Q \in GL_s(\mathbb{Q})$. Prove that

$$\ker(PA) = \ker(A), \qquad \ker(AQ) = Q^{-1} \cdot \ker(A),$$
$$\mathrm{im}(PA) = P \cdot \mathrm{im}(A), \qquad \mathrm{im}(AQ) = \mathrm{im}(A).$$

43. Let $A \in M_{t \times s}(\mathbb{Q})$. Show that $\ker(A)$ is a subspace of $\mathbb{Q}^s$.

44. Let $A \in M_{t \times s}(\mathbb{Q})$. Show that $\mathrm{im}(A)$ is a subspace of $\mathbb{Q}^t$.

45. Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q}).$$

Let

$$A = P1_rQ.$$

Prove that

$$\ker(A) \text{ has basis } \{\text{last } s-r \text{ columns of } Q^{-1}\},$$

$$\operatorname{im}(A) \text{ has basis } \{\text{first } r \text{ columns of } P\}.$$

46. Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q}).$$

Let

$$A = P1_rQ.$$

Prove that

$$\dim(\operatorname{im}(A)) + \dim(\ker(A)) = (\text{number of columns of } A).$$

47. Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q}).$$

Let

$$A = P1_rQ.$$

Prove that $\dim(\operatorname{im}(A)) = \operatorname{rank}(A)$.

48. Let $A \in M_{t\times s}(\mathbb{Q})$. Show that there exist $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$ and $r \in \{1, \ldots, \min(s,t)\}$ such that

$$A = P1_rQ.$$

where $1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q})$.

49. Let

$$1_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis of $\ker(1_2)$.

(b) Find a basis of $\operatorname{im}(1_2)$.

50. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}$.

(a) Show that $A = P1_2Q$, where

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Compute Q^{-1}.

(c) Find a basis of $\ker(A)$.

(d) Find a basis of $\mathrm{im}(A)$.

51. (Column space) Let $A \in M_{t\times s}(\mathbb{Q})$ and write

$$A = P1_rQ, \qquad \text{with } P \in GL_t(\mathbb{Q}),\ Q \in GL_s(\mathbb{Q}) \text{ and } r \in \{1, \ldots, \min(s,t)\}.$$

The *image*, or *column space*, of A is

$$\mathrm{im}(A) = \mathrm{span}\{c_1, \ldots, c_n\}, \quad \text{where } c_1, \ldots, c_n \in \mathbb{R}^m \text{ are the columns of } A.$$

(a) Show that
$$\mathrm{im}(A) = \mathrm{span}\{\text{first } r \text{ columns of } P\}.$$

(b) Show that the first r columns of P form a basis of $\mathrm{im}(A)$.

52. (Other formulations of rank) Let $A \in M_{t\times s}(\mathbb{Q})$ and write

$$A = P1_rQ, \qquad \text{with } P \in GL_t(\mathbb{Q}),\ Q \in GL_s(\mathbb{Q}) \text{ and } r \in \{1, \ldots, \min(s,t)\}.$$

The *row space* of A is

$$\mathrm{rowsp}(A) = \mathrm{span}\{r_1, \ldots, r_m\}, \quad \text{where } r_1, \ldots, r_m \in \mathbb{R}^n \text{ are the rows of } A.$$

(a) Show that the row space of A is $\mathrm{im}(A^t)^t$.

(b) Use that $A^t = Q^t 1_r^t P^t$ to show that
$$\mathrm{rowsp}(A) = \mathrm{span}\{\text{first } r \text{ rows of } Q\}.$$

(c) Show that the first r rows of Q form a basis of $\mathrm{rowsp}(A)$.

53. (Basis of the null space by row reduction) Let $A \in M_{t\times s}(\mathbb{Q})$ and write

$$A = P1_rQ, \qquad \text{with } P \in GL_t(\mathbb{Q}),\ Q \in GL_s(\mathbb{Q}) \text{ and } r \in \{1, \ldots, \min(s,t)\}.$$

The *null space* of A is $\ker(A)$.

(a) Show that
$$\ker(A) = \mathrm{span}\{\text{last } s-r \text{ columns of } Q^{-1}\}.$$

(b) Show that, if $P \in \mathrm{GL}_4(\mathbb{Q})$ and

$$A = P \begin{pmatrix} 1 & 7 & 0 & 6 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{then} \quad \left\{ \begin{pmatrix} 7 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\} \quad \text{is a basis of } \ker(A).$$

54. Let

$$R = \begin{pmatrix} 0 & 0 & 1 & 8 & 3 & 0 & 9 & 1 & 3 & 4 & 0 & 2 & 6 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 2 & 2 & 8 & 0 & 4 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 3 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in M_{7\times 15}(\mathbb{Q}).$$

Let

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & -3 & -9 & -1 & -3 & -4 & -2 & -6 & -5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & -2 & -2 & -8 & -4 & -1 & -7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -3 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in M_{15\times(15-4)}(\mathbb{Q}).$$

Show that $\ker(R) = \mathrm{im}(S)$.

55. (Rank and dimension of column space and null space) Let A be an $m \times n$ matrix. The *kernel*, or *null space*, of A is

$$\ker(A) = \mathrm{nullsp}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}.$$

The *image*, or *column space*, of A is

$$\mathrm{im}(A) = \mathrm{span}\{c_1, \ldots, c_n\}, \quad \text{where } c_1, \ldots, c_n \in \mathbb{R}^m \text{ are the columns of } A.$$

The *rank* of A is $\dim(\mathrm{im}(A))$ and the *nullity* of A is $\dim(\ker(A))$.

Show that

$$(\text{number of columns of } A) - \dim(\mathrm{im}(A)) = \dim(\ker(A)).$$

56. (Other formulations of rank) Assume $A \in M_{m\times n}(\mathbb{F})$. The *row space* of A is

$$\mathrm{rowsp}(A) = \mathrm{span}\{r_1, \ldots, r_m\}, \quad \text{where } r_1, \ldots, r_m \in \mathbb{R}^n \text{ are the rows of } A.$$

The *column space*, of A is

$$\mathrm{im}(A) = \mathrm{span}\{c_1, \ldots, c_n\}, \quad \text{where } c_1, \ldots, c_n \in \mathbb{R}^m \text{ are the columns of } A.$$

Show that

$$\mathrm{rank}(A) = \dim(\mathrm{rowsp}(A)) = \dim(\mathrm{colsp}(A)).$$

57. Assume $A \in M_{m\times n}(\mathbb{Q})$ and that $P \in GL_m(\mathbb{Q})$ and $R \in M_{m\times n}(\mathbb{Q})$ are such that R is in reduced row echelon form and $A = PR$. Show that

$$\mathrm{rank}(A) = (\text{number of nonzero rows in the row echelon form of } A).$$

58. (Basis of the kernel by row reduction) Suppose that $A \in M_{4\times 6}(\mathbb{R})$ so that A is a 4×6 matrix. Suppose that $A = PR$, where $P \in GL_4(\mathbb{R})$ and

$$R = \begin{pmatrix} 1 & 7 & 0 & 6 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Show that

$$\left\{ \begin{pmatrix} 7 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 3 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\} \qquad \text{is a basis of } \ker(A).$$

59. (Basis of the column space by row reduction) Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$. Create the $n \times k$ matrix A with $v_1, v_2, \ldots, v_k$ as columns. Let $P \in GL_n(\mathbb{R})$ and $R \in M_{n\times k}(\mathbb{R})$ in reduced row echelon form such that $A = PR$. A pivot of R is an entry of R which is the first nonzero entry in its row. Show that if $j_1, j_2, \ldots, j_\ell$ are the columns where the pivots of R are then $v_{j_1}, v_{j_2}, \ldots, v_{j_\ell}$ is a basis of $\mathrm{span}\{v_1, \ldots, v_k\}$.

60. (Basis of the row space by row reduction) Let $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$. Create the $k \times n$ matrix A^t with $v_1, v_2, \ldots, v_k$ as rows. Let $P \in GL_k(\mathbb{R})$ and $R \in M_{k\times n}(\mathbb{R})$ be in reduced row echelon form such that $A^t = PR$. Show that if $b_{i_1}, b_{i_2}, \ldots, b_{i_\ell}$ are the nonzero rows of R are then $b_{i_1}, b_{i_2}, \ldots, b_{i_\ell}$ is a basis of $\mathrm{span}\{v_1, \ldots, v_k\}$.

4 Eigenvectors, eigenvalues and diagonalization

4.1 Eigenvalues and eigenvectors

1. Let $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Find $P, D \in GL_2(\mathbb{Q})$ such that D is diagonal and $A = PDP^{-1}$.

2. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ viewed as an element of $M_{2\times 2}(\mathbb{R})$. Show that A does not have any eigenvalues or eigenvectors. Conclude that A is not diagonalizable as an element of $M_{2\times 2}(\mathbb{R})$.

3. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ viewed as an element of $M_{2\times 2}(\mathbb{C})$.

   (a) Find the eigenvalues and eigenvectors of A.

   (b) Find $P, D \in GL_2(\mathbb{C})$ such that D is diagonal and $A = PDP^{-1}$.

   Conclude that A is diagonalizable as an element of $M_{2\times 2}(\mathbb{C})$.

4. Find a matrix $A \in M_{2\times 2}(\mathbb{C})$ that does not have 2 linearly independent eigenvectors. Conclude that A is not diagonalizable.

5. Let $A = \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A acting on $\mathbb{Q}^2$.

6. Let $A = \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A acting on $\mathbb{R}^2$.

7. Let $A = \begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A acting on $\mathbb{C}^2$.

8. Let $A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A acting on $\mathbb{Q}^3$.

9. Let $A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A acting on $\mathbb{R}^3$.

10. Let $A = \begin{pmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{pmatrix}$.

    (a) Find the eigenvalues and eigenvectors of A.

    (b) Find $P, D \in GL_2(\mathbb{Q})$ such that D is diagonal and $A = PDP^{-1}$.

11. Let $A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A acting on $\mathbb{C}^3$.

12. Find the eigenvalues and eigenvectors of
$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} \qquad \text{acting on } \mathbb{C}^3.$$

13. Find the eigenvalues and eigenvectors of
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix} \qquad \text{acting on } \mathbb{C}^3.$$

14. Find the eigenvalues and eigenvectors of
$$A = \begin{pmatrix} 2 & i & 1+2i \\ -i & 0 & -i \\ 1-2i & i & 0 \end{pmatrix} \qquad \text{acting on } \mathbb{C}^3.$$

15. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A acting on $\mathbb{Q}^5$.

16. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A acting on $\mathbb{R}^5$.

17. Let $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$. Find the eigenvalues and eigenvectors of A acting on $\mathbb{C}^5$.

18. Find the eigenvalues and eigenvectors of
$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \qquad \text{acting on } \mathbb{C}^4.$$

19. Find the eigenvalues and eigenvectors of
$$A = \begin{pmatrix} 3 & 2 & 2 & -4 \\ 2 & 3 & 2 & -1 \\ 1 & 1 & 2 & -1 \\ 2 & 2 & 2 & -1 \end{pmatrix} \qquad \text{acting on } \mathbb{C}^4.$$

20. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & -1 & -1 \\ -2 & 0 & -2 & -3 & -2 \\ 2 & 0 & 2 & 4 & 3 \end{pmatrix} \qquad \text{acting on } \mathbb{C}^5.$$

21. Let $A, B \in M_{n\times n}(\mathbb{R})$. Prove that AB and BA have the same eigenvalues.

22. Let $A \in M_{n\times n}(\mathbb{R})$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A.

    (a) Show that if A is invertible then $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$ are the eigenvalues of A^{-1}.

    (b) Let $k \in \mathbb{Z}_{>0}$. Show that $\lambda_1^k, \ldots, \lambda^k$ are the eigenvalues of A^k.

    (c) Let $k \in \mathbb{Z}_{>0}$. What are the eigenvectors of A^k?

23. Let $A \in M_{n\times n}(\mathbb{R})$. Prove that A and A^t have the same eigenvalues.

4.2 Diagonalization

24. Let $D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Compute D^{10}.

25. Assume $A = PDP^{-1}$. Prove that if $k \in \mathbb{Z}$ then $A^k = PD^kP^{-1}$.

26. Let

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}.$$

    (a) Compute P^{-1} and prove that $A = PDP^{-1}$.

    (b) Compute A^k for $k \in \mathbb{Z}$.

27. Let

$$x_n = \begin{pmatrix} r_n \\ p_n \\ w_n \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

    (a) Show that

$$\text{if} \quad P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{then} \quad P^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

    (b) Show that $T = PDP^{-1}$.

    (c) Let $r_0, p_0, w_0 \in \mathbb{R}$ and let

$$x_0 = \begin{pmatrix} r_0 \\ p_0 \\ w_0 \end{pmatrix}. \qquad \text{Compute } \lim_{n\to\infty} T^n x_0.$$

    (d) Explain how this is used in applications to genetics.

28. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

29. Let $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

30. Let $A = \begin{pmatrix} -2 & -8 & -12 \\ 1 & 4 & 4 \\ 0 & 0 & 1 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

31. Let $A = \begin{pmatrix} 2-i & 0 & i \\ 0 & 1+i & 0 \\ i & 0 & 2-i \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

32. Let $A = \begin{pmatrix} -1 & -1 & -6 & 3 \\ 1 & -2 & -3 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & -5 & 3 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

33. Let $A = \begin{pmatrix} 3 & -1 & -2 & 2 \\ 3 & -1 & -2 & 2 \\ 2 & -1 & -1 & 2 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

34. Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

35. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 6 & -3 \\ 5 & -2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 6 & -3 \\ 5 & -2 \end{pmatrix} \qquad \text{acting on } \mathbb{Q}^2.$$

   (a) Show that A is conjugate to a diagonal matrix and explain why B is not.

   (b) Find an invertible matrix $P \in GL_2(\mathbb{Q})$ such that $P^{-1}AP$ is diagonal.

36. Let $\alpha \in \mathbb{C}$. Determine the eigenvalues of $A = \begin{pmatrix} 1 & \alpha & \alpha^2 & 0 \\ 1 & \alpha & 0 & \alpha^2 \\ 1 & 0 & \alpha & \alpha^2 \\ 0 & 1 & \alpha & \alpha^2 \end{pmatrix}$. Find an invertible matrix $P \in GL_n(\mathbb{C})$ such that $P^{-1}AP$ is diagonal.

37. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

   (a) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

   (b) Let $n \in \mathbb{Z}_{>0}$. Find A^n.

38. Let $A = \begin{pmatrix} -1 & 4 & -2 \\ 1 & -1 & 1 \\ 3 & -6 & 4 \end{pmatrix}$.

    (a) Find an invertible matrix P such that $P^{-1}AP$ is diagonal.

    (b) Find a matrix X such that $X^2 = A$ and $X \neq A$ and $X \neq -A$.

39. Let $A \in M_n(\mathbb{F})$. Prove that the matrix A has n linearly independent eigenvectors $p_1, \ldots, p_n \in \mathbb{F}^n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if $A = PDP^{-1}$ where,
$$P = \begin{pmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{pmatrix} \qquad \text{and} \qquad D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$
so that $p_1, \ldots, p_n$ are the columns of P and D is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.

40. Assume that the inner product on $\mathbb{F}^n$ is the standard inner product. Let $A \in M_n(\mathbb{F})$. The matrix A has n orthonormal eigenvectors $q_1, \ldots, q_n \in \mathbb{F}^n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if $A = QDQ^{-1}$ and $QQ^t = 1$ where,
$$Q = \begin{pmatrix} | & & | \\ q_1 & \cdots & q_n \\ | & & | \end{pmatrix} \qquad \text{and} \qquad D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$
so that $u_1, \ldots, u_n$ are the columns of Q and D is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.

41. Assume that the inner product on $\mathbb{C}^n$ is the standard Hermitian inner product. Let $A \in M_n(\mathbb{C})$. The matrix A has n orthonormal eigenvectors $u_1, \ldots, u_n \in \mathbb{C}^n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if $A = UDU^{-1}$ and $U\overline{U}^t = 1$ where,
$$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \qquad \text{and} \qquad D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$
so that $u_1, \ldots, u_n$ are the columns of U and D is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.

4.3 Symmetric, hermitian, orthogonal and unitary matrices

42. Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Find an orthogonal matrix $Q \in O_2(\mathbb{R})$ and a diagonal matrix $D \in M_{2\times 2}(\mathbb{R})$ such that $A = QDQ^T$.

43. Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Find an unitary matrix $Q \in U_2(\mathbb{C})$ and a diagonal matrix $D \in M_{2\times 2}(\mathbb{C})$ such that $A = QDQ^T$.

44. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Find $U \in O_2(\mathbb{R})$, $V \in O_3(\mathbb{R})$ and $S \in M_{2\times 3}(\mathbb{R})$ such that S is diagonal and $A = USV^T$.

45. Let $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. Find $U \in O_2(\mathbb{R})$, $V \in O_2(\mathbb{R})$ and $S \in M_{2\times 2}(\mathbb{R})$ such that S is diagonal and $A = USV^T$.

46. Let $A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{pmatrix}$. Find an orthogonal matrix U such that $U^t AU$ is a diagonal matrix.

47. Let $A = \begin{pmatrix} 11 & 0 & 6 \\ 0 & 5 & 6 \\ 6 & 6 & -2 \end{pmatrix}$. Find an orthogonal matrix U such that $U^t AU$ is a diagonal matrix.

48. Let $A = \begin{pmatrix} 7 & -1 & -2 \\ -1 & 7 & 2 \\ -2 & 2 & 10 \end{pmatrix}$. Find an orthogonal matrix U such that $U^t AU$ is a diagonal matrix.

49. Let $A = \begin{pmatrix} 18 & -1 & -4 \\ -1 & 18 & -4 \\ -4 & -4 & 3 \end{pmatrix}$. Find an orthogonal matrix U such that $U^t AU$ is a diagonal matrix.

50. Reduce the following real quadratic forms to diagonal form.

    (i) $x^2 + y^2 + xy$

    (ii) $x^2 + y^2 - xy$

    (iii) $3x^2 + 3y^2 + 5xz - 2xy$

    (iv) $2xy + 2xz + 2yz$

    (v) $5x^2 + 11y^2 - 2x^2 + 12xz + 12yz$

51. Find the principal axes, centre and sketch the graph of the following conics.

    (i) $xy = 2$

    (ii) $3x^2 - 2y^2 + 12xy = 42$

5 Vector spaces and linear transformations

5.1 Subspaces

1. (a) Is the line $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x + 1\}$ a subspace of $\mathbb{R}^2$?
   (b) Is the line $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x\}$ a subspace of $\mathbb{R}^2$?

2. Let $T: V \to W$ be an $\mathbb{R}$-linear transformation.
   Show that $\ker(T) = \{v \in V \mid T(v) = 0\}$ is a subspace of V.

3. Let $T: V \to W$ be an $\mathbb{R}$-linear transformation.
   Show that $\mathrm{im}(T) = \{T(v) \mid v \in V\}$ is a subspace of W.

4. (Subspaces of $\mathbb{R}^2$)
   (a) Show that a line through the origin is a subspace of $\mathbb{R}^2$.
   (b) Show that a line that does not go through the origin is not a subspace of $\mathbb{R}^2$.
   (c) Show that every subspace of $\mathbb{R}^2$ that is not 0 and not $\mathbb{R}^2$ is a line through the origin.

5. (Subspaces of $\mathbb{R}^3$)
   (a) Show that a line through the origin is a subspace of $\mathbb{R}^3$.
   (b) Show that a plane through the origin is a subspace of $\mathbb{R}^3$.
   (c) Show that a line that does not go through the origin is not a subspace of $\mathbb{R}^3$.
   (d) Show that a plane that does not go through the origin is not a subspace of $\mathbb{R}^3$.
   (e) Show that if W is a subspace of $\mathbb{R}^3$ then either $W = 0$ or W is a line through the origin or W is a plane through the origin or $W = \mathbb{R}^3$.

6. Let $n \in \mathbb{Z}_{\geq 3}$. Determine (with proof)
$$\text{whether} \qquad \{|\alpha_1, \ldots, \alpha_n\rangle \mid \alpha_1 + \alpha_2 + \alpha_3 = 0\} \qquad \text{is a subspace of } \mathbb{R}^n.$$

7. Let $n \in \mathbb{Z}_{\geq 3}$. Determine (with proof)
$$\text{whether} \qquad \{|\alpha_1, \ldots, \alpha_n\rangle \mid \alpha_1 + \alpha_2 + \alpha_3 = 1\} \qquad \text{is a subspace of } \mathbb{R}^n.$$

8. Let $n \in \mathbb{Z}_{\geq 3}$. Determine (with proof)
$$\text{whether} \qquad \{|\alpha_1, \ldots, \alpha_n\rangle \mid \alpha_3 = \alpha_1\alpha_2\} \qquad \text{is a subspace of } \mathbb{R}^n.$$

9. Is $W = \{|x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}$ a $\mathbb{R}$-subspace of $\mathbb{R}^3$?

10. Is $W = \{a_1x + a_2x^2 \mid a_1, a_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}[x]_{\leq 2}$?

11. Is the set of real 2×2 matrices whose trace is equal to 0 a subspace of $M_{2\times 2}(\mathbb{R})$?

12. Is
$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \mid ad - bc = 0 \right\} \text{ a subspace of } M_2(\mathbb{R})?.$$

13. Let $n \in \mathbb{Z}_{\geq 3}$. Determine (with proof)
$$\text{whether} \qquad \{|\alpha_1, \ldots, \alpha_n\rangle \mid |\alpha_1| > 0\} \qquad \text{is a subspace of } \mathbb{R}^n.$$

14. Let $n \in \mathbb{Z}_{\geq 3}$. Determine (with proof)
$$\text{whether} \qquad \{|\alpha_1, \ldots, \alpha_n\rangle \mid \alpha_1 - \alpha_2 = \alpha_2 - \alpha_3\} \qquad \text{is a subspace of } \mathbb{R}^n.$$

15. Let V be the set of sequences $(\alpha_n) = (\alpha_0, \alpha_1, \ldots)$ of elements of $\mathbb{R}$. Define addition of sequences by
$$(\alpha_0, \alpha_1, \alpha_2, \ldots) + (\beta_0, \beta_1, \beta_2, \ldots) = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$$
and scalar multiplication by
$$c(\alpha_0, \alpha_1, \alpha_2, \ldots) = (c\alpha_0, c\alpha_1, c\alpha_2, \ldots), \qquad \text{for } c \in \mathbb{R}.$$
Prove that V is an $\mathbb{R}$-vector space.

16. Prove that the following are $\mathbb{R}$-vector spaces.
    (a) The set of arithmetic progressions:
    $$\{(\alpha_n) \mid \alpha_i \in \mathbb{R} \text{ and if } n \in \mathbb{Z}_{\geq 0} \text{ then } \alpha_{n+1} - \alpha_n = \alpha_{n+1} - \alpha_{n+1}\}.$$
    (b) The set of Fibonacci sequences:
    $$\{(\alpha_n) \mid \alpha_i \in \mathbb{R} \text{ and if } n \in \mathbb{Z}_{\geq 0} \text{ then } \alpha_{n+2} = \alpha_{n+1} + \alpha_n\}.$$
    (c) The set of convergent sequences:
    $$\left\{ (\alpha_n) \;\middle|\; \begin{array}{c} \alpha_i \in \mathbb{R} \text{ and there exists } \alpha \in \mathbb{R} \text{ such that} \\ \text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \\ \text{if } n \in \mathbb{Z}_{>N} \text{ then } |\alpha_n - \alpha| < \epsilon \end{array} \right\}.$$
    (d) The set of Cauchy sequences:
    $$\left\{ (\alpha_n) \;\middle|\; \begin{array}{c} \alpha_i \in \mathbb{R} \text{ and there exists } \alpha \in \mathbb{R} \text{ such that} \\ \text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } N \in \mathbb{Z}_{>0} \text{ such that} \\ \text{if } m, n \in \mathbb{Z}_{>N} \text{ then } |\alpha_n - \alpha_m| < \epsilon \end{array} \right\}.$$

17. Let $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{\text{functions } f \colon \mathbb{R} \to \mathbb{R}\}$. Determine (with proof)
$$\text{whether} \qquad \text{the set of real polynomials of degree exactly } n \qquad \text{is a subspace of } \mathcal{F}(\mathbb{R}, \mathbb{R}).$$

18. Let $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{\text{functions } f \colon \mathbb{R} \to \mathbb{R}\}$. Determine (with proof)
$$\text{whether} \qquad \text{the set of real polynomials of degree } < n \qquad \text{is a subspace of } \mathcal{F}(\mathbb{R}, \mathbb{R}).$$

19. Let $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \{\text{functions } f \colon \mathbb{R} \to \mathbb{R}\}$. An *odd function* is a function $f \colon \mathbb{R} \to \mathbb{R}$ such that if $x \in \mathbb{R}$ then $f(-x) = -f(x)$. Determine (with proof)
$$\text{whether} \qquad \text{the set of odd functions} \qquad \text{is a subspace of } \mathcal{F}(\mathbb{R}, \mathbb{R}).$$

20. Let $\mathcal{F}(\mathbb{R},\mathbb{R}) = \{\text{functions } f\colon \mathbb{R}\to\mathbb{R}\}$. Determine (with proof)
$$\text{whether} \qquad \text{the set of differentiable functions} \qquad \text{is a subspace of } \mathcal{F}(\mathbb{R},\mathbb{R}).$$

21. Let $C(\mathbb{R}_{[0,1]},\mathbb{R})$ be the set of continuous functions $f\colon \mathbb{R}_{[0,1]}\to\mathbb{R}$. Determine (with proof)
$$\text{whether} \qquad \{f\in C(\mathbb{R}_{[0,1]},\mathbb{R}) \mid f(1)=0\} \qquad \text{is a subspace of } C(\mathbb{R}_{[0,1]},\mathbb{R}).$$

22. Let $C(\mathbb{R}_{[0,1]},\mathbb{R})$ be the set of continuous functions $f\colon \mathbb{R}_{[0,1]}\to\mathbb{R}$. Determine (with proof)
$$\text{whether} \qquad \{f\in C(\mathbb{R}_{[0,1]},\mathbb{R}) \mid f(1)=1\} \qquad \text{is a subspace of } C(\mathbb{R}_{[0,1]},\mathbb{R}).$$

23. Let $C(\mathbb{R}_{[0,1]},\mathbb{R})$ be the set of continuous functions $f\colon \mathbb{R}_{[0,1]}\to\mathbb{R}$. Determine (with proof)
$$\text{whether} \qquad \{f\in C(\mathbb{R}_{[0,1]},\mathbb{R}) \mid f(0)=f(1)\} \qquad \text{is a subspace of } C(\mathbb{R}_{[0,1]},\mathbb{R}).$$

24. Let $C(\mathbb{R}_{[0,1]},\mathbb{R})$ be the set of continuous functions $f\colon \mathbb{R}_{[0,1]}\to\mathbb{R}$. Determine (with proof)
$$\text{whether} \qquad \Big\{f\in C(\mathbb{R}_{[0,1]},\mathbb{R}) \;\Big|\; \int_0^1 f(t)dt=0\Big\} \qquad \text{is a subspace of } C(\mathbb{R}_{[0,1]},\mathbb{R}).$$

25. Let $C(\mathbb{R}_{[0,1]},\mathbb{R})$ be the set of continuous functions $f\colon \mathbb{R}_{[0,1]}\to\mathbb{R}$. Determine (with proof)
$$\text{whether} \qquad \Big\{f\in C(\mathbb{R}_{[0,1]},\mathbb{R}) \;\Big|\; \int_0^1 f(t)dt=1\Big\} \qquad \text{is a subspace of } C(\mathbb{R}_{[0,1]},\mathbb{R}).$$

26. Determine (with proof)
$$\text{whether} \qquad \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{R}) \;\middle|\; a=b\right\} \qquad \text{is a subspace of } M_2(\mathbb{R}).$$

27. Determine (with proof)
$$\text{whether} \qquad \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{R}) \;\middle|\; a+b=1\right\} \qquad \text{is a subspace of } M_2(\mathbb{R}).$$

28. Determine (with proof)
$$\text{whether} \qquad \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{R}) \;\middle|\; a=c=d\right\} \qquad \text{is a subspace of } M_2(\mathbb{R}).$$

29. Determine (with proof)
$$\text{whether} \qquad \{A\in M_2(\mathbb{R}) \mid \det(A)=0\} \qquad \text{is a subspace of } M_2(\mathbb{R}).$$

30. Determine (with proof)
$$\text{whether} \qquad \{A\in M_2(\mathbb{R}) \mid \det(A)=1\} \qquad \text{is a subspace of } M_2(\mathbb{R}).$$

31. Let $B \in M_2(\mathbb{R})$. Determine (with proof)

$$\text{whether} \qquad \{A \in M_2(\mathbb{R}) \mid AB = BA\} \qquad \text{is a subspace of } M_2(\mathbb{R}).$$

32. Let $B \in M_2(\mathbb{R})$. Determine (with proof)

$$\text{whether} \qquad \{A \in M_2(\mathbb{R}) \mid A^2 = A\} \qquad \text{is a subspace of } M_2(\mathbb{R}).$$

33. Determine (with proof)

$$\text{whether} \qquad \{A \in M_3(\mathbb{R}) \mid \det(A) = 0\} \qquad \text{is a subspace of } M_3(\mathbb{R}).$$

34. Determine (with proof)

$$\text{whether} \qquad \{A \in M_3(\mathbb{R}) \mid \mathrm{tr}(A) = 0\} \qquad \text{is a subspace of } M_3(\mathbb{R}).$$

35. Determine (with proof)

$$\text{whether} \qquad \{A \in M_3(\mathbb{R}) \mid A^t = -A\} \qquad \text{is a subspace of } M_3(\mathbb{R}).$$

5.2 Linear transformations

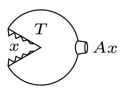
36. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.
Let $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ be the function given by $T(x) = Ax$ so that

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ 5x_5 + 6x_6 + 7x_7 + 8x_8 \end{pmatrix}.$$

Show that T is a linear transformation.

37. Let $t, s \in \mathbb{Z}_{>0}$ and $A \in M_{t\times s}(\mathbb{R})$. Let $T\colon \mathbb{R}^s \to \mathbb{R}^t$ be the function given by

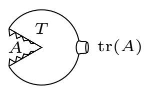
$$T(x) = Ax.$$



Show that T is a linear transformation.

38. Let $n \in \mathbb{Z}_{>0}$ and let $T\colon M_n(\mathbb{Q}) \to \mathbb{Q}$ by the function given by

$$T(A) = \mathrm{tr}(A).$$



Show that T is a $\mathbb{Q}$-linear transformation.

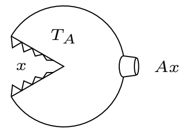
39. Let $T\colon M_{2\times 2}(\mathbb{Q}) \to \mathbb{Q}$ be the function given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Show that T is a linear transformation.

40. Let $t, s \in \mathbb{Z}_{>0}$ and $A \in M_{t\times s}(\mathbb{R})$. Let $T_A\colon \mathbb{R}^s \to \mathbb{R}^t$ be the function given by

$$T_A(x) = Ax.$$



Show that T_A is a linear transformation.

41. Let $T\colon V \to W$ be a linear transformation. Assume that T has an inverse function $T^{-1}\colon W \to V$. Show that T^{-1} is a linear transformation.

42. Is the function $T\colon M_2(\mathbb{R}) \to \mathbb{R}$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \quad \text{a linear transformation?}$$

43. Is the function $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3) = (x_2 - 2x_3, 3x_1 + x_3) \qquad \text{a linear transformation?}$$

44. Let X and Y be $\mathbb{R}$-vector spaces. Let $T\colon X \to Y$ be a function such that if $x_1, x_2 \in X$ then $T(x_1 + x_2) = T(x_1) + T(x_2)$. Show that $T(0) = 0$.

45. Let X and Y be $\mathbb{R}$-vector spaces. Let $T\colon X \to Y$ be a function such that if $x_1, x_2 \in X$ then $T(x_1 + x_2) = T(x_1) + T(x_2)$. Show that if $x \in X$ then $T(-x) = -t(x)$.

46. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function given by

$$T(|\alpha, \beta, \gamma\rangle) = |\alpha + \beta - \gamma, 2\alpha + \beta\rangle.$$

Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

47. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function given by

$$T(|\alpha, \beta, \gamma\rangle) = |\alpha + 1, \alpha + 2\beta - \gamma\rangle.$$

Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

48. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function given by

$$T(|\alpha, \beta, \gamma\rangle) = |\ |\alpha|, 0\rangle.$$

Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

49. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function given by

$$T(|\alpha, \beta, \gamma\rangle) = |\alpha\beta, \beta\alpha\rangle.$$

Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

50. Let $S \in M_{n\times n}(\mathbb{R})$. Let $T\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be the function given by

$$T(A) = AS.$$

Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

51. Let $S \in M_{n\times n}(\mathbb{R})$. Let $T\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be the function given by
$$T(A) = AS - SA.$$
Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

52. Let $T\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be the function given by
$$T(A) = A^t.$$
Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

53. Let $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$ be the function given by
$$T(f) = \frac{df}{dx}.$$
Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

54. Let $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$ be the function given by
$$T(f) = \int_{t=0}^{t=x} f(t)dt.$$
Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

55. Let $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$ be the function given by
$$T(f) = f \cdot \frac{df}{dx}.$$
Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

56. Let $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$ be the function given by
$$T(f) = xf.$$
Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

57. Let $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$ be the function given by
$$T(f(x)) = f(x+1).$$
Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

58. Let $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$ be the function given by
$$T(f(x)) = \frac{d^2f}{dx^2} - 2\frac{df}{dx} + 3.$$
Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

59. Let $a, b \in \mathbb{R}[x]$. Let $T\colon \mathbb{R}[x] \to \mathbb{R}[x]$ be the function given by
$$T(f(x)) = \frac{d^2f}{dx^2} + a\frac{df}{dx} + bf.$$
Determine (with proof) whether T is an $\mathbb{R}$-linear transformation.

60. Let $T: V \to W$ be a linear transformation. Assume that the inverse function $T^{-1}: W \to V$ exists. Show that T^{-1} is a linear transformation.

61. Let $S: V \to W$ and $T: V \to W$ be $\mathbb{R}$-linear transformations. Define a function $(S+T): V \to W$ by
$$(S+T)(v) = S(v) + T(v).$$
Show that $(S+T)$ is a $\mathbb{R}$-linear transformation.

62. Let $T: V \to W$ be a $\mathbb{R}$-linear transformation and let $c \in \mathbb{R}$. Define a function $(cT): V \to W$ by
$$(cT)(v) = c \cdot T(v).$$
Show that (cT) is a $\mathbb{R}$-linear transformation.

63. Let V and W be $\mathbb{R}$-vector spaces and let
$$X = \{T: V \to W \mid T \text{ is a linear transformation}\}.$$
Show that X with the functions
$$\begin{array}{ccc} X \times X & \to X & \\ (S,T) & \to & (S+T) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{R} \times X & to & X \\ (c,T) & \mapsto & (cT) \end{array}$$
given by
$$(S+T)(v) = S(v) + T(v) \qquad \text{and} \qquad (cT)(v) = c \cdot T(v)$$
is a vector space.

6 Linear independence, span and bases

1. Let V be a $\mathbb{Q}$-vector space and let $v_1, v_2, v_3, v_4, v_5 \in V$. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$. Show that $\mathbb{Q}$-span(S) is a subspace of V.

2. In $\mathbb{R}[x]_{\le 2}$, is $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$?

3. Let S be the subset of $\mathbb{R}[x]_{\le 2}$ given by
$$S = \{1 + 2x, 1 + 5x + 3x^2, x + x^2\}. \qquad \text{Show that } \mathbb{R}\text{-span}(S) = \mathbb{R}[x]_{\le 2}.$$

4. In $\mathbb{R}^3$, is $|1, 2, 3\rangle \in \mathbb{R}\text{-span}\{|1, -1, 2\rangle, |-1, 1, 2\rangle\}$?

5. In $\mathbb{R}[x]_{\le 2}$, is $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$?

6. Let S be the subset of $\mathbb{R}^3$ given by
$$S = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}. \qquad \text{Determine } \mathbb{R}\text{-span}(S).$$

7. Let S be the subset of $\mathbb{R}^2$ given by
$$S = \{|1, -1\rangle, |2, 4\rangle\}. \qquad \text{Show that } \mathbb{R}\text{-span}(S) = \mathbb{R}^2.$$

8. Let S be the subset of $\mathbb{R}^3$ given by
$$S = \{|1, 2, 0\rangle, |1, 5, 3\rangle, |0, 1, 1\rangle\}. \qquad \text{Show that span}(S) = \mathbb{R}^3.$$

9. Let S be the subset of $\mathbb{R}[x]_{\le 2}$ given by
$$S = \{1 + x + x^2, x^2\}. \qquad \text{Show that span}(S) = \mathbb{R}[x]_{\le 2}.$$

10. Let S be the subset of $\mathbb{C}^3$ given by
$$S = \{|2i, -1, 1\rangle, |-6, -3i, 3i\rangle\}. \qquad \text{Is } S\ \mathbb{C}\text{-linearly independent?}$$

11. Let B be the subset of $\mathbb{C}^3$ given by
$$B = \{|2i, -1, 1\rangle, |4, 0, 2\rangle\}. \qquad \text{Is } B \text{ linearly independent?}$$

12. Let S be the subset of $\mathbb{R}^3$ given by
$$S = \{(2, 0, 0), (6, 1, 7), (2, -1, 2)\}. \qquad \text{Is } S \text{ linearly independent?}$$

13. Let S be the subset of $M_2(\mathbb{R})$ given by
$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\}. \qquad \text{Is } S \text{ linearly independent?}$$

14. Let S be the subset of $\mathbb{R}[x]_{\le 2}$ given by
$$S = \{1 + 2x + 5x^2, 1 + x + x^2, 1 + 2x + 3x^2\}. \quad \text{Is } S \text{ a basis of } \mathbb{R}[x]_{\le 2}?$$

15. Is $S = \{(1,-1),(2,4)\}$ a basis of $\mathbb{R}^2$?

16. Is $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ a basis of $\{A \in M_2(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$?

17. Is
$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \qquad \text{a basis of } M_2(\mathbb{R})?$$

18. Is $S = \{(1,-1),(2,4)\}$ a basis of $\mathbb{R}^2$?

19. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ be given by
$$v_1 = |1,2,3\rangle, \quad v_2 = |3,6,9\rangle, \quad v_3 = |-1,0,-2\rangle, \quad v_4 = |1,4,4\rangle.$$
   (a) Is $\{v_1, v_2, v_3, v_4\}$ linearly independent?
   (b) Express v_2 and v_4 as linear combinations of v_1 and v_3.
   (c) Is $\{v_1, v_3\}$ linearly independent?

20. Let S be the subset of $M_2(\mathbb{R})$ given by
$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\}. \qquad \text{Is } S \text{ linearly independent?}$$

21. Prove that $\{|1,0,3\rangle, |5,2,1\rangle, |0,1,6\rangle\}$ is an $\mathbb{R}$-basis for $\mathbb{R}^3$.

22. Let $t \in \mathbb{R}$ and let
$$g_1(x) = 1, \qquad g_2(x) = x + t, \qquad g_3(x) = (x+t)^2.$$
Prove that $\{g_1, g_2, g_3\}$ is an $\mathbb{R}$-basis of $\mathbb{R}[x]_{\le 2}$.

23. (a) Prove that $B = \{|1,2,0\rangle, |0,5,7\rangle, |-1,1,3\rangle\}$ is an $\mathbb{R}$-basis for $\mathbb{R}^3$.
   (b) Find the coordinates of $|0,13,17\rangle$ and $|2,3,1\rangle$ relative to the basis B.

24. (a) Show that $S = \{|3+\sqrt{2}, 1+\sqrt{2}\rangle, |7, 1+2\sqrt{2}\rangle\}$ is $\mathbb{R}$-linearly dependent.
   (b) Show that $S = \{|3+\sqrt{2}, 1+\sqrt{2}\rangle, |7, 1+2\sqrt{2}\rangle\}$ is $\mathbb{Q}$-linearly independent.

25. (a) Show that $S = \{|1-i, i\rangle, |2, -1+i\rangle\}$ is $\mathbb{C}$-linearly dependent.
   (b) Show that $S = \{|1-i, i\rangle, |2, -1+i\rangle\}$ is $\mathbb{R}$-linearly independent.

26. Show that $S = \{e^t, \sin(t), t^2\}$ is $\mathbb{R}$-linearly independent.

27. Show that $S = \{e^t, \sin(t), \cos(t)\}$ is $\mathbb{R}$-linearly independent.

28. (a) Show that $S = \{|1,0,1,1\rangle, |1,0,2,4\rangle\}$ is $\mathbb{R}$-linearly dependent.
   (b) Extend S to a basis of $\mathbb{R}^4$.

29. Let V be a $\mathbb{C}$-vector space and let $u, v, w \in V$. Assume that $\{u, v, w\}$ is $\mathbb{C}$-linearly independent.
   (a) Show that $S = \{u+v, v+w, w+u\}$ is $\mathbb{R}$-linearly independent.
   (b) Show that $S = \{u+v-3w, u+3v-w, v+w\}$ is $\mathbb{C}$-linearly dependent.

30. (a) Show that $S = \{|1,1,0,-1\rangle, |4,-2,1,0\rangle\}$ is $\mathbb{Q}$-linearly independent.

(b) Find a $\mathbb{Q}$-basis for $\mathbb{Q}^4$ containing S.

31. Let M and N be the subspaces of $M_2(\mathbb{R})$ given by

$$M = \left\{ \begin{pmatrix} a & b \\ -b & c \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\} \qquad \text{and} \qquad N = \left\{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}.$$

Find $\mathbb{R}$-bases for M, N, $M \cap N$ and $M + N$.

32. Let S and T be the subspaces of $\mathbb{R}^4$ given by

$$S = \{|\alpha, \beta, \gamma, \delta\rangle \mid \alpha + \beta + \gamma = 0\} \qquad \text{and} \qquad T = \{|\alpha, \beta, \gamma, \delta\rangle \mid \gamma = -\delta\}.$$

Find $\mathbb{R}$-bases for S, T, $S \cap T$ and $S + T$.

33. Find an $\mathbb{R}$-basis for the solution space of the equation $x - 2y + z - 3t = 0$.

34. Let V_1 and V_2 be 2-dimensional subspaces of $\mathbb{R}^3$. Prove that $\dim(V_1 \cap V_2) > 0$.

35. Let U be the subspace of $\mathbb{R}^2$ generated by $u_1 = |1, 2, 3\rangle$ and $u_2 = |3, -5, 1\rangle$.
    (a) Show that $|1, 0, 0\rangle$ is not in U but that $|5, -23, -9\rangle$ is in U.
    (b) Express $|5, -23, -9\rangle$ as a linear combination of u_1 and u_2.

36. Is $S = \{1, 1+t, 1+t+t^2, \ldots, 1+t+\cdots+t^n\}$ a basis of $\mathbb{R}[t]_{\le n}$?

37. Is $S = \{1, 1+t, t+t^2, \ldots, t^{n-1}+t^n\}$ a basis of $\mathbb{R}[t]_{\le n}$?

38. Is $S = \{1, 1-t, (1-t)^2, \ldots, (1-t)^n\}$ a basis of $\mathbb{R}[t]_{\le n}$?

39. Let $u = |3, -1, 0, -1\rangle$ and $v = |1, 0, 4, -1\rangle$ in $\mathbb{R}^4$ and let $S = \{|2, -1, 3, 2\rangle, |-1, 1, 1, -3\rangle, |1, 1, 9, -5\rangle\}$.
    (a) Are u and v in $\mathrm{span}(S)$.
    (b) Find a basis of $\mathrm{span}(S)$ containing u.
    (c) Find a basis of $\mathrm{span}(S)$ containing v.

40. Let V be an $\mathbb{R}$-vector space. Assume that S is a linearly independent subset of V and let $v \in V$. Show that $v \notin \mathrm{span}(S)$ if and only if $S \cup \{v\}$ is linearly independent.

41. Let $S = \{|1, 0, -1, 1\rangle, |2, -1, 0, 1\rangle, |1, 1, 2, 1\rangle\}$ in $\mathbb{R}^4$.
    (a) Show that S is linearly independent.
    (b) Show that $|1, 3, 3, 2\rangle \in \mathrm{span}(S)$.
    (c) Show that $|0, 1, 1, -1\rangle \notin \mathrm{span}(S)$.
    (d) Find an $\mathbb{R}$-basis for $\mathrm{span}(S)$ which contains $|1, 3, 3, 2\rangle$.
    (e) Find an $\mathbb{R}$-basis for $\mathbb{R}^4$ which contains $|0, 1, 1, -1\rangle$.

42. Let $V = M_{2\times 2}(\mathbb{C})$.
    (a) Explain how to view V as a $\mathbb{C}$-vector space and find a $\mathbb{C}$-basis of V.
    (b) Explain how to view V as a $\mathbb{R}$-vector space and find a $\mathbb{R}$-basis of V.

43. Let $V = M_{2\times 2}(\mathbb{C})$ and let

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right\}.$$

(a) Find a $\mathbb{C}$-basis of $\mathbb{C}$-span(S).

(b) Find a $\mathbb{R}$-basis of $\mathbb{R}$-span(S).

44. Let $S = \{|1,1,0\rangle, |i, 1+i, 1\rangle, |1+i, 1+i, 0\rangle\}$ and $T = \{|1,0,1\rangle, |i, -i, 0\rangle, |0, i, i\rangle$ in $\mathbb{C}^3$.

(a) Find a $\mathbb{C}$-basis of $\mathbb{C}$-span$(S) \cap \mathbb{C}$-soan(T).

(b) Find a $\mathbb{C}$-basis of $\mathbb{C}$-span$(S) + \mathbb{C}$-soan(T).

(c) Show that $\mathbb{R}$-span$(S) \cap \mathbb{R}$-soan$(T) = \{0\}$.

45. Let $S = \{1 - t^2 + t^3, 2 + t - t^2 + t^3, 1 + 2t + t^2 - t^3\}$ in $\mathbb{R}[t]_{\le 3}$.

(a) Show that $t + t^2 - t^3 \in \text{span}(S)$.

(b) Show that $1 + t - t^2 + t^3 \notin \text{span}(S)$.

(c) Find an $\mathbb{R}$-basis for span(S) that contains $t + t^2 - t^3$.

(c) Find an $\mathbb{R}$-basis for $\mathbb{R}[t]_{\le 3}$ that contains $1 + t - t^2 + t^3$.

46. (basis of $\mathbb{F}^n$) Let $\mathbb{F}$ be a field and let

$$\mathbb{F}^n = \{\ 1 \times n \text{ matrices with entries in } \mathbb{F}\}.$$

Let

e_i be the $1 \times n$ matrix with 1 in the ith entry and all other entries 0.

Show that $\{e_1, e_2, \ldots, e_n\}$ is a basis of $\mathbb{F}^n$.

47. (basis of $M_{m\times n}(\mathbb{F})$) Let $\mathbb{F}$ be a field and let

$$M_{m\times n}(\mathbb{F}) = \{\ m \times n \text{ matrices with entries in } \mathbb{F}\}.$$

Let

E_{ij} be the $m \times n$ matrix with 1 in the $(i.j)$ entry and all other entries 0

Show that $\{E_{ij} \mid i \in \{1, \ldots, m\} \text{ and } j \in \{1, \ldots, n\}\}$ is a basis of $M_{m\times n}(\mathbb{F})$.

48. (basis of $\mathbb{F}[x]$) Let $\mathbb{F}$ be a field and let

$$\mathbb{F}[x] = \{\text{polynomials with coefficients in } \mathbb{F}\}.$$

Show that $\{1, x, x^2, x^3, \ldots\}$ is a basis of $\mathbb{F}[x]$.

49. (basis of $\mathbb{F}[x]_{\le d}$) Let $d \in \mathbb{Z}_{\ge 0}$ and let $\mathbb{F}$ be a field. Let

$$\mathbb{F}[x]_{\le d} = \{\text{polynomials of degrees} \le d \text{ with coefficients in } \mathbb{F}\}.$$

Show that $\{1, x, x^2, x^3, \ldots, x^d\}$ is a basis of $\mathbb{F}[x]_{\le d}$.

50. ($\mathbb{R}$ as a $\mathbb{Q}$ vector space) Find a basis of $\mathbb{R}$ as $\mathbb{Q}$-vector space.

7 Working with linear transformations

7.1 Matrix of a linear transformation with respect to bases B and C

1. Let $v = (1, 5) \in \mathbb{R}^2$.

   (a) Find the coordinate vector of v with respect to the basis $S = \{(1, 0), (0, 1)\}$.

   (b) Find the coordinate vector of v with respect to the basis $B = \{(2, 1), (-1, 1)\}$.

2. Find the coordinate vector of $p = 2 + 7x - 9x^2$ with respect to the basis $B = \{2, \frac{1}{2}x, -3x^2\}$ of $\mathbb{Q}[x]_{\le 2}$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the function given by $T(x_1, x_2, x_3) = |x_2 - 2x_3, 3x_1 + x_3\rangle$.

   (a) Prove that T is a linear transformation.

   (b) Find the matrix of T with respect to the basis $S = \{|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle\}$ of $\mathbb{R}^3$ and the basis $B = \{|1, 0\rangle, |0, 1\rangle\}$ of $\mathbb{R}^2$.

4. Let $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by
$$T(Q) = Q^t.$$
   Find the matrix of T with respect to the basis $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, where E_{ij} is the matrix with 1 in the (i, j) entry and 0 elsewhere.

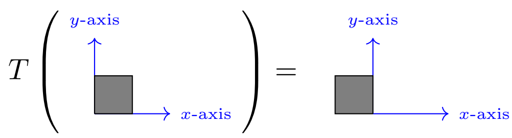
5. Let $T : \mathbb{R}[x]_{\le 2} \to \mathbb{R}[x]_{\le 1}$ be the linear transformation given by
$$T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_2) + a_0 x.$$

   (a) Find the matrix of T with respect to the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$ and the basis $C = \{1, x\}$ of $\mathbb{R}[x]_{\le 1}$.

   (b) Find the matrix of T with respect to the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$ and the basis $D = \{2, 3x\}$ of $\mathbb{R}[x]_{\le 1}$.

6. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and that the matrix of T with respect to the basis $A = \{|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle\}$ of $\mathbb{R}^3$ and the basis $S = \{|1, 0\rangle, |0, 1\rangle\}$ of $\mathbb{R}^2$ is
$$[T]_{SA} = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix}.$$
   Find the matrix of T with respect to the basis $B = \{|1, 1, 0\rangle, |1, -1, 0\rangle, |1, -1, -2\rangle\}$ of $\mathbb{R}^3$ and the basis $C = \{|1, 1\rangle, |1, -1\rangle\}$ of $\mathbb{R}^2$.

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which is reflection across the y-axis.



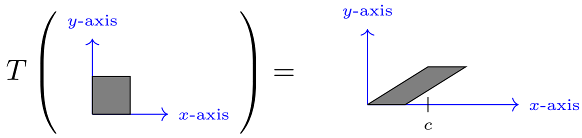
   Find the matrix of T with respect to the basis $S = \{|1, 0\rangle, |0, 1\rangle\}$.

8. Find the matrix of the linear transformation

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{which is projection onto the } x \text{ axis.}$$

Is T injective? Is T surjective. Is T invertible?

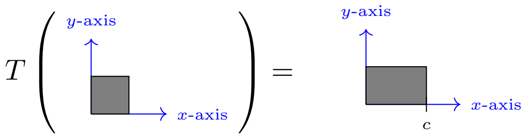
9. Let $c \in \mathbb{R}_{>0}$ and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is the shear by a factor of c along the x-axis.



Find the matrix of T with respect to the basis $S = \{|1,0\rangle, |0,1\rangle\}$.

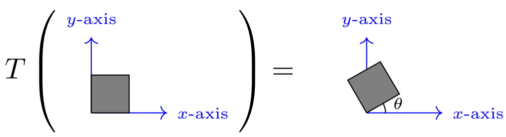
10. Find the image of $(x, y) \in \mathbb{R}^2$ after a shear along the x-axis with $c = 1$ followed by a reflection across the y-axis.

11. Let $c \in \mathbb{R}_{>0}$ and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is stretching of the x-axis by a factor of c.



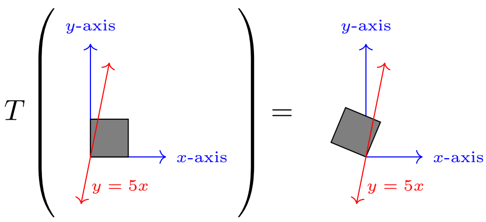
Find the matrix of T with respect to the basis $B = \{(1,0), (0,1)\}$.

12. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is rotation by θ (about the origin counter-clockwise).



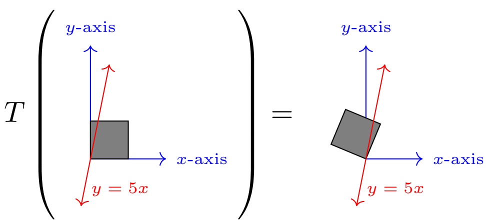
Find the matrix of T with respect to the basis $S = \{(1,0), (0,1)\}$.

13. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $y = 5x$.



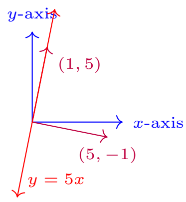
Identify two lines through the origin that are invariant under T and find the image of the direction vectors for each of these lines.

14. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is reflection in the line $y = 5x$.



Let $S = \{(1,0),(0,1)\}$ and let

$$B = \{(1,5),(5,-1)\},$$



so that the first vector in B is a vector in the direction of the line $y = 5x$ and the second vector in B is a vector perpendicular to $y = 5x$.

(a) Compute $[T]_{BB}$, $[I]_{SB}$, $[I]_{BS}$, and $[T]_{SS}$.

(b) Compute $T\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix}$.

15. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x,y) = (3x - y, -x + 3y).$$

and let B and S be the bases of $\mathbb{R}^2$ given by

$$B = \{(1,1),(-1,1)\} \qquad \text{and} \qquad S = \{(1,0),(0,1)\}.$$

Let u and v be the vectors in $\mathbb{R}^2$ determined by

$$[u]_B = \begin{pmatrix}1\\1\end{pmatrix} \qquad \text{and} \qquad [v]_S = \begin{pmatrix}0\\2\end{pmatrix}.$$

(a) Compute $[I]_{BS}$, and $[I]_{SB}$

(b) Compute $[u]_S$ and $[v]_B$.

(c) Compute $[T]_{SS}$ and $[T]_{BB}$.

16. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T(|x,y,z\rangle) = |x - y, x + 2y - z, 2x + y + z\rangle.$$

Find the matrix of T with respect to the standard basis of $\mathbb{R}^3$.

17. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by

$$T(|x,y,z\rangle) = |x - y, x + 2y - z, 2x + y + z\rangle.$$

Find the matrix of T with respect to the basis $B = \{|1,0,1\rangle, |-2,1,1\rangle, |1,-1,1\rangle\}$ of $\mathbb{R}^3$.

18. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that the matrix of T with respect to the standard basis is

$$\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}.$$

Find the matrix of T with respect to the basis $B = \{|1,0,1\rangle, |-2,1,1\rangle, |1,-1,1\rangle\}$ of $\mathbb{R}^3$.

19. Let $T\colon \mathbb{R}[x]_{\le n} \to \mathbb{R}[x]_{\le n}$ be the linear transformation given by
$$T(f) = \frac{df}{dx}.$$
Find the matrix of T with respect to the basis $B = \{1, x, x^2, \ldots, x^n\}$ of $\mathbb{R}[x]_{\le n}$.

20. Let $T\colon \mathbb{R}[x]_{\le n} \to \mathbb{R}[x]_{\le n}$ be the linear transformation given by
$$T(f) = \frac{df}{dx}.$$
Find the matrix of T with respect to the basis $B = \{1, x-1, (x-1)^2, \ldots, (x-1)^n\}$ of $\mathbb{R}[x]_{\le n}$.

21. Let $T\colon \mathbb{R}[x]_{\le n} \to \mathbb{R}[x]_{\le n}$ be the linear transformation given by
$$T(f) = \frac{df}{dx}.$$
Find the matrix of T with respect to the basis $B = \{1, 1+x, 1+x+x^2, \ldots, 1+x+\cdots+x^n\}$ of $\mathbb{R}[x]_{\le n}$.

22. Let $T\colon \mathbb{R}[x]_{\le n} \to \mathbb{R}[x]_{\le n}$ be the linear transformation given by
$$T(f(x)) = f(x+1).$$
Find the matrix of T with respect to the basis $B = \{1, x, x^2, \ldots, x^n\}$ of $\mathbb{R}[x]_{\le n}$.

23. Let $T\colon \mathbb{R}[x]_{\le n} \to \mathbb{R}[x]_{\le n}$ be the linear transformation given by
$$T(f(x)) = f(x+1).$$
Find the matrix of T with respect to the basis $B = \{1, x-1, (x-1)^2, \ldots, (x-1)^n\}$ of $\mathbb{R}[x]_{\le n}$.

24. Let $T\colon \mathbb{R}[x]_{\le n} \to \mathbb{R}[x]_{\le n}$ be the linear transformation given by
$$T(f(x)) = f(x+1).$$
Find the matrix of T with respect to the basis $B = \{1, 1+x, 1+x+x^2, \ldots, 1+x+\cdots+x^n\}$ of $\mathbb{R}[x]_{\le n}$.

25. Let $S \in M_{n\times n}(\mathbb{R})$. Let $T\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be the linear transformation given by
$$T(A) = SA.$$
Find the matrix of T with respect to the basis $B = \{E_{ij} \mid i, j \in \{1, \ldots, n\}\}$ of $M_{n\times n}(\mathbb{R})$.

26. Let $S \in M_{n\times n}(\mathbb{R})$. Let $T\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be the linear transformation given by
$$T(A) = AS.$$
Find the matrix of T with respect to the basis $B = \{E_{ij} \mid i, j \in \{1, \ldots, n\}\}$ of $M_{n\times n}(\mathbb{R})$.

27. Let $S \in M_{n\times n}(\mathbb{R})$. Let $T\colon M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ be the linear transformation given by
$$T(A) = SA - AS.$$
Find the matrix of T with respect to the basis $B = \{E_{ij} \mid i, j \in \{1, \ldots, n\}\}$ of $M_{n\times n}(\mathbb{R})$.

28. Let $B = \{u_1, u_2\}$ be a basis of $\mathbb{R}^2$ and let $X = \{v_1, v_2, v_3\}$ be a basis of $\mathbb{R}^3$. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$\begin{aligned} T(u_1) &= v_1 + 2v_2 - v_3, \\ T(u_1) &= v_1 - v_2. \end{aligned}$$

   (a) Find the matrix of T with respect to the basis B of $\mathbb{R}^2$ and the basis X of $\mathbb{R}^3$.
   (b) Let $C = \{-u_1 + u_2, 2u_1 - u_2\}$ and $Y = \{v_1, v_1 + v_2, v_1 + v_2 + v_3\}$. Find the matrix of T with respect to the basis C of $\mathbb{R}^2$ and the basis Y of $\mathbb{R}^3$.

29. Let $\lambda, \mu \in \mathbb{R}$. Let U be a $\mathbb{R}$-vector space with basis $B = \{u_1, u_2, u_1\}$ and let V be a $\mathbb{R}$-vector space with basis $X = \{v_1, v_2, v_3\}$. Let $T\colon U \to V$ be a linear transformation such that the matrix of T with respect to the bases B and X is

$$[T]_{XB} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \lambda & \mu \\ 1 & \lambda^2 & \mu^2 \end{pmatrix}.$$

Let $C = \{u_1 + u_2 + u_3, u_1 + (\lambda + 1)u_3, u_3\}$.

   (a) Find the matrix of T with respect to the bases C and X.
   (b) Find the values of λ and μ for which the system

$$\begin{aligned} x + y + z &= 1, \\ x + \lambda y + \mu z &= 2, \\ x + \lambda^2 y + \mu^2 z&, \end{aligned} \qquad \text{has a unique solution.}$$

   (c) Find the values of λ and μ for which the system

$$\begin{aligned} x + y + z &= 1, \\ x + \lambda y + \mu z &= 2, \\ x + \lambda^2 y + \mu^2 z&, \end{aligned} \qquad \text{has more than one solution.}$$

30. The vector space of homogeneous real quandratic polynomials in x, y, z is

$$V = \{a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz \mid a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}\}.$$

Let $\alpha, \beta, \gamma \in \mathbb{R}$ and let $T\colon V \to V$ be the linear transformation given by

$$T(f(x, y, z)) = f(\alpha x + y + z, \beta y + \gamma z, 0).$$

Find the matrix of t with respect to the basis $B = \{x^1, y^2, z^2, xy, xz, yz\}$ of V.

31. The vector space of real quadratic polynomials in x and y is

$$U = \{a_1x^2 + a_2y^2 + a_3xy + a_4x + a_5y + a_6 \mid a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}\}.$$

The vector space of real cubic polynomials in x is

$$V = \{c_1x^3 + c_2x^2 + c_3x + a_4 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\}.$$

Let $T\colon U \to V$ be the function given by

$$T(f(x, y)) = f(x, 2).$$

(a) Show that T is a linear transformation.
(b) Find the matrix of T with respect to the basis $B = \{x^2, y^2, xy, x, y, 1\}$ of U and the basis $C = \{x^3, x^2, x, 1\}$ of V.

32. Carefully define the matrix of a linear transformation with respect to bases B and C.

33. Carefully define the change of basis matrix from B to C.

34. Prove the following proposition.

Proposition 7.1. *Let V and W and Z be $\mathbb{F}$-vector spaces with bases B, C and D, respectively. Let*
$$f\colon V \to W, \quad g\colon V \to W, \quad h\colon W \to Z \quad \text{be } \mathbb{F}\text{-linear transformations}$$
and let $c \in \mathbb{F}$. Then
$$(cf)_{CB} = c \cdot f_{CB}, \qquad f_{CB} + g_{CB} = (f+g)_{CB} \qquad \text{and} \qquad (h \circ g)_{DB} = h_{DC} g_{CB}.$$

35. Prove the following proposition.

Proposition 7.2. *Let $g\colon V \to W$ and $f\colon V \to V$ be an $\mathbb{F}$-linear transformations. Let*
$$B_1 \text{ and } B_2 \text{ be bases of } V, \quad \text{and let} \quad C_1 \text{ and } C_2 \text{ be bases of } W,$$
*and let $P_{B_1B_2}$ and $P_{C_2C_1}$ be the change of basis matrices defined as in (**??**). Then*
$$g_{C_2B_2} = P_{C_2C_1} g_{C_1B_1} P_{B_1B_2} \qquad \text{and} \qquad f_{B_2B_2} = P^{-1}_{B_1B_2} f_{B_1B_1} P_{B_1B_2}.$$

7.2 Kernels and images of linear transformations

36. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by
$$T(x, y, z) = (2x - y, y + z).$$
Find bases for $\ker(T)$ and $\mathrm{im}(T)$ and verify the rank-nullity theorem.

37. Let $T\colon \mathbb{R}[x]_{\le 2} \to \mathbb{R}[x]_{\le 1}$ be the linear transformation given by
$$T(a_0 + a_1x + a_2x^2) = (a_0 - a_1 + a_2)(1 + 2x).$$
(a) Find bases for $\ker(T)$ and $\mathrm{im}(T)$.
(b) Is T injective?
(c) Is T surjective?

38. The derivative with respect to x is the linear transformation $T\colon \mathbb{R}[x]_{\le 3} \to \mathbb{R}[x]_{\le 2}$ given by
$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$
(a) Find the matrix of T with respect to the basis $S = \{1, x, x^2, x^3\}$ of $\mathbb{R}[x]_{\le 3}$ and the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$.
(b) Find $\ker(T)$ and $\mathrm{im}(T)$.

39. Let $T\colon V \to W$ be a linear transformation.

(a) Define $\ker(T)$.
(b) Define $\mathrm{im}(T)$.
(a) Show that $\ker(T)$ is a subspace of V.
(b) Show that $\mathrm{im}(T)$ is a subspace of W.

40. Let $T\colon X \to Y$ be a linear transformation.
    (a) Define $\ker(T)$.
    (b) Define $\mathrm{im}(T)$.
    (c) Define what it means for T to be injective.
    (d) Define what it means for T to be surjective.
    (e) Prove that T is injective if and only if $\ker(T) = 0$.
    (f) Prove that T is surjective if and only if $\mathrm{im}(T) = Y$.

41. (transpose) Show that the function
$$T\colon \begin{array}{ccc} M_{r\times c}(\mathbb{F}) & \to & M_{c\times r}(\mathbb{F}) \\ A & \mapsto & A^t \end{array} \qquad \text{is an } \mathbb{F}\text{-linear transformation.}$$
Find the matrix of T respect to the basis $\{E_{ij} \mid i \in \{1,\ldots,t\}, j \in \{1,\ldots,s\}\}$. Determine $\ker(T)$ and $\mathrm{im}(T)$.

42. (derivative) Show that the function $f\colon \mathbb{C}[x] \to \mathbb{C}[x]$ given by
$$f(p) = \frac{dp}{dx} \qquad \text{is an } \mathbb{F}\text{-linear transformation.}$$
Find the matrix of f with respect to the basis $\{1, x, x^2, \ldots\}$. Determine $\ker(f)$ and $\mathrm{im}(f)$.

43. (matrices give linear transformations) Let $V = \mathbb{R}^{35}$ and let $A \in M_{35\times 35}(\mathbb{R})$. Show that the function
$$T\colon V \to V \qquad \text{given by} \quad T(v) = Av$$
is a linear transformation.

44. (more matrix linear transformations) Let $V = M_{3\times 2}(\mathbb{R})$ and let $A \in M_{3\times 3}(\mathbb{R})$.
    (a) Show that the function $T\colon V \to V$ given by
$$T(v) = Av \qquad \text{is a linear transformation.}$$
    (b) Find the matrix of T with respect to the basis $\{E_{11}, E_{12}, E_{21}, E_{22}, E_{31}, E_{32}\}$.

45. (determinant is not a linear transformation) Let $V = M_{13\times 13}(\mathbb{R})$. Show that the function
$$T\colon V \to \mathbb{R} \qquad \text{given by} \quad T(V) = \det(v),$$
is not a linear transformation.

46. (rotations, reflections, stretchers, shears and projections) Draw pictures of the linear transformations $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by multiplication the matrices
$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

47. Let V be an $\mathbb{R}$-vector space and let $\{b_1, \ldots, b_{12}\}$ be a basis of V. Show that the function

$$\begin{array}{cccc} T\colon & V & \to & \mathbb{R}^{12} \\ & c_1v_1 + \cdots + c_{12}v_{12} & \mapsto & \begin{pmatrix} c_1 \\ \vdots \\ c_{12} \end{pmatrix} \end{array}$$ is a bijective linear transformation.

48. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(|\alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle) = |\alpha_1 - \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4, 3\alpha_2 + 2\alpha_3\rangle.$$

Find bases for $\ker(T)$ and $\mathrm{im}(T)$.

49. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation with matrix relative to the standard bases given by

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 2 & 6 & 3 & -3 \\ 0 & 2 & 5 & -7 \end{pmatrix}.$$

Find the dimension of $\ker(T)$ and find the dimension of $\mathrm{im}(T)$.

50. Define a linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^3$ by

$$T(|\alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle) = |\alpha_1 - \alpha_3 + 2\alpha_4, -2\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_2 + 4\alpha_4\rangle.$$

(a) Find the dimension of $\ker(T)$ and find the dimension of $\mathrm{im}(T)$.

(b) Show that $|1, 3, k\rangle$ is in $\mathrm{im}(T)$ if and only if $k = 5$.

(c) Find the condition for $|1, x, 1, y\rangle$ to be in $\ker(T)$.

51. Define functions $S\colon \mathbb{R}[x,y]_{\le n} \to \mathbb{R}[x,y]_{\le n}$ and $T\colon \mathbb{R}[x,y]_{\le n} \to \mathbb{R}[x,y]_{\le n}$ by

$$S(f) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \qquad \text{and} \qquad T(f) = x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2}.$$

(a) Show that S and T are $\mathbb{R}$-linear transformations.

(b) Find $\ker(S)$, $\mathrm{im}(S)$, $\ker(T)$ and $\mathrm{im}(T)$.

(c) Specify an $\mathbb{R}$-basis of $\mathbb{R}[x,y]_{\le n}$ and find the matrices of S and T with respect to your specified basis.

52. Let $D\colon \mathbb{R}[[x]] \to \mathbb{R}[[x]]$ be the linear transformation given by $D(f) = \frac{df}{dx}$.

(a) Find $\ker(D)$.

(b) Let $n \in \mathbb{Z}_{>0}$. Find $\ker(D^n)$.

(c) Find $\ker(D - 1)$.

53. Let $S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(A) = SA.$$

Find bases for $\ker(T)$ and $\mathrm{im}(T)$.

54. Let $S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by
$$T(A) = AS.$$
Find bases for $\ker(T)$ and $\mathrm{im}(T)$.

55. Let $S = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by
$$T(A) = SA - AS.$$
Find bases for $\ker(T)$ and $\mathrm{im}(T)$.

56. Let $T\colon V \to W$ be a linear transformation and let $U \subseteq W$ be a subspace of W. Show that
$$X = \{v \in V \mid T(v) \in U\} \qquad \text{is a subspace of } V$$
and find the dimension of X in terms of $\dim(V)$, $\dim(W)$, $\dim(U)$ and $\dim(\mathrm{im}(T))$.

57. Let $S\colon V \to V$ and $T\colon V \to V$ be linear transformations. Prove that
$$\dim(\ker(ST)) \le \dim(\ker(S)) + \dim(\ker(T)).$$

58. Let $n \in \mathbb{Z}_{>0}$ and let $S\colon V \to V$ be a linear transformation such that $S^n = 0$ and $S^{n-1} \neq 0$. Determine $\dim(\ker(S))$.

59. Let $S\colon U \to V$ and $T\colon V \to U$ be linear transformations. Prove that
$$\dim(\mathrm{im}(T)) - \dim(\ker(ST)) \le \dim(U) - \dim(\mathrm{im}(S)).$$

60. Give an example of a linear transformation $T\colon V \to V$ such that $\ker(T) = \mathrm{im}(T)$.

61. Give an example of vector space V such that there does not exist a linear transformation $T\colon V \to V$ such that $\ker(T) = \mathrm{im}(T)$.

62. Let $T\colon V \to V$ be a linear transformation.
    (a) Assume that if $v \in V$ and $T(T(v)) = 0$ then $T(v) = 0$. Show that $\mathrm{im}(T) \cap \ker(T) = \{0\}$.
    (b) Assume that $\mathrm{im}(T) \cap \ker(T) = \{0\}$. Show that if $v \in V$ and $T(T(v)) = 0$ then $T(v) = 0$.

63. Let $T\colon V \to V$ be a linear transformation such that $\dim(\mathrm{im}(T^2)) = \dim(\mathrm{im}(T))$. Show that $\mathrm{im}(T) \cap \ker(T) = \{0\}$.

64. Let $T\colon V \to V$ be a linear transformation such that $T^2 = T$.
    (a) Show that $\ker(T) = \mathrm{im}(1-T)$ and $\ker(1-T) = \mathrm{im}(T)$.
    (b) Show that $\ker(T) \cap \mathrm{im}(T) = \{0\}$.
    (c) Show that if $v \in V$ then there exists unique $v_1 \in \ker(T)$ and $v_2 \in \mathrm{im}(T)$ such that $v = v_1 + v_2$.

7.3 Invertible linear transformations

65. Let V be an $\mathbb{R}$-vector space and let $\{v_1, v_2, v_3, v_4\}$ be a basis of V. Let $\lambda \in \mathbb{R}$ and let $T\colon V \to V$ be the $\mathbb{R}$-linear transformation given by
$$T(v_1) = v_1 + \lambda v_4, \qquad T(v_2) = 2v_1 + v_2, \qquad T(v_3) = 2v_2 + v_3, \qquad T(v_4) = 2v_3 + v_2.$$
Determine (with proof) $\{\lambda \in \mathbb{R} \mid T \text{ is invertible}\}$.

66. Let $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ be the $\mathbb{R}$-linear transformation defined by
$$T(|\alpha_1, \alpha_2, \alpha_3\rangle) = |3\alpha_1 - \alpha_2, \alpha_1 - \alpha_2 + \alpha_3, -\alpha_1 + 2\alpha_2 - \alpha_3\rangle.$$
   (a) Show that T is invertible.
   (b) Compute $T^{-1}(|\beta_1, \beta_2, \beta_3\rangle)$.

67. Let $T\colon \mathbb{C} \to \mathbb{C}$ be the $\mathbb{R}$-linear transformation defined by $T(z) = (1 - i)z$. Show that T is invertible.

68. Let $n \in \mathbb{Z}_{\geq 0}$. Let $D\colon \mathbb{R}[x]_{\leq n} \to \mathbb{R}[x]_{\leq n}$ be the linear transformation given by $D(f) = \dfrac{df}{dx}$.
   (a) Show that D is not invertible.
   (b) Determine $\ker(D)$.

69. Let V be a $\mathbb{C}$-vector space and let $T\colon V \to V$ be a $\mathbb{C}$-linear transformation such that $T^2 = 0$. Show that $1 - T$ is invertible.

70. Let $T\colon V \to V$ be a linear transformation. Show that T is invertible if and only if $\ker(T) = 0$ and $\mathrm{im}(T) = V$.

71. Let $T\colon V \to V$ be a linear transformation. Show that if V is finite dimensional and $\ker(T) = 0$ then $\mathrm{im}(T) = V$.

72. Let $T\colon V \to V$ be a linear transformation. Show that if V is finite dimensional $\mathrm{im}(T) = V$ then $\ker(T) = 0$.

7.4 The rank of a linear transformation

73. The vector space of homogeneous real quandratic polynomials in x, y, z is
$$V = \{a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz \mid a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{R}\}.$$
Let $\alpha, \beta, \gamma \in \mathbb{R}$ and let $T\colon V \to V$ be the linear transformation given by
$$T(f(x, y, z)) = f(\alpha x + y + z, \beta y + \gamma z, 0).$$
Find the rank of T.

74. Let $S \in M_n(\mathbb{Q})$. Define a $\mathbb{Q}$-linear transformation
$$T\colon M_n(\mathbb{Q}) \to M_n(\mathbb{Q}) \qquad \text{by} \qquad T(A) = AS.$$
   (a) Assume that S is invertible. Show that the rank of T is n^2.
   (b) Show that if $S \in M_n(\mathbb{Q})$ then $\mathrm{rank}(T) = n \cdot \mathrm{rank}(S)$.

7.5 Eigenvalues and eigenvectors of linear transformations

75. Let $T\colon \mathbb{R}[x]_{\le n} \to \mathbb{R}[x]_{\le n}$ be the linear transformation given by $T(f) = \frac{df}{dx}$. Find the eigenvalues and eigenvectors of T.

76. Find the eigenvalues and eigenvectors of reflections and rotations in $\mathbb{R}^2$.

8 Inner products

8.1 Inner products

1. (The standard inner product on $\mathbb{C}^n$) Show that

$$\langle x, y\rangle = x_1\overline{y_1} + \cdots + x_n\overline{y_n},$$

defnes an inner product on $\mathbb{C}^n$.

2. (The standard inner product on $M_{m\times n}(\mathbb{F})$) Show that

$$\langle A, B\rangle = \mathrm{tr}(A^t B).$$

defines an inner product on $M_{m\times n}(\mathbb{F})$.

3. (The standard inner product on polynomials) Show that

$$\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx.$$

defines an inner product on the vector space $\mathbb{C}[x]$.

4. Let $u = |1+i, 1 = i\rangle$ and $v = |i, 1\rangle$ in $\mathbb{C}^2$ with the standard inner product. Compute $\langle u|v\rangle$, $\langle v|u\rangle$, $\langle u|u\rangle$, $\langle u|u\rangle$, $d(u, v)$, $\cos(\theta(u, v))$ and $\theta(u, v)$.

5. Let $u = 1$ and $v = x$ in $\mathbb{R}[x]_{\le 2}$ with the standard inner product. Compute $\langle u|u\rangle$, $\langle v|v\rangle$, $\langle u|v\rangle$, $d(u, v)$, $\cos(\theta(u, v))$, and $\theta(u, v)$.

6. Let $A \in M_{n\times n}(\mathbb{C})$ satisifying $A = \bar{A}^t$ and let $\langle,\rangle\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be given by

$$\langle (u_1, \ldots, u_n), (v_1, \ldots, v_n)\rangle = (u_1, \ldots, u_n)A\begin{pmatrix}\overline{v_1}\\ \vdots \\ \overline{v_n}\end{pmatrix} = u^t A\bar{v},$$

Prove that $\langle,\rangle$ satisfies all the properties of an inner product, except perhaps the positive definiteness.

7. Let $u, v \in \mathbb{R}^n$ and let $Q \in O_n(\mathbb{R})$. Prove that $\langle Qu|Qv\rangle = \langle u|v\rangle$.

8. Is the map $\langle,\rangle\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle (u_1, u_2, u_3), (v_1, v_2, v_3)\rangle = u_1v_1 - u_2v_2 + u_3v_3 \qquad \text{positive definite?}$$

9. Is the map $\langle,\rangle\colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$ given by

$$\langle (u_1, u_2), (v_1, v_2)\rangle = iu_1\overline{v_1} - iu_2\overline{v_2} \qquad \text{positive definite?}$$

10. Is the map $\langle,\rangle\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\langle (u_1, u_2), (v_1, v_2)\rangle = 2u_1v_1 - 2u_1v_2 - 2u_2v_1 + u_2v_2 \qquad \text{positive definite?}$$

11. Is the map $\langle,\rangle\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by
$$\langle (u_1, u_2), (v_1, v_2)\rangle = u_1v_1 + 2u_2v_2 \qquad \text{positive definite?}$$

12. (Pythagorean theorem) Let V be an $\mathbb{F}$-vector space with an inner product. Let $u, v \in V$ and suppose that u and v are orthogonal. Prove that $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

13. Let V be a $\mathbb{F}$-vector space with an inner product $\langle,\rangle\colon V \times V \to \mathbb{F}$.
Let $B = \{b_1, \ldots, b_n\}$ be a basis of V and let A be the Gram matrix of $\langle,\rangle$ with respect to the basis B. Let $u, v \in V$. Prove that
$$\langle u, v\rangle = [u]_B^t A[\bar{v}]_B.$$

14. Let $\langle\ ,\ \rangle\colon \mathbb{R}[x]_{\le 2} \times \mathbb{R}[x]_{\le 2} \to \mathbb{R}$ be the standard inner product on the $\mathbb{R}$-vector space $\mathbb{R}[x]_{\le 2}$.
    (a) Let $u = 7 + 3x + 2x^2$ and $v = 5 + x^2$. Calculate $\langle u|v\rangle$.
    (b) Let A be the Gram matrix of $\langle\ ,\ \rangle$ with respect to the basis $B = \{1, x, x^2\}$. Calculate A.
    (c) Calculate $[u]_B$ and $[v]_B$ and $[u]_B^t A[v]_B$.

15. Let $\langle,\rangle\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by
$$\langle (u_1, u_2), (v_1, v_2)\rangle = u_1v_1 + 2u_2v_2 = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$
Let $u = |3, 1\rangle$ and $v = |-2, 3\rangle$. Find $\|u\|$, $d(u, v)$ and $\theta(u, v)$.

16. Are $|2, -1, 1\rangle$ and $|1, 2, 1\rangle$ perpendicular in $\mathbb{R}^3$ with the standard inner product?

17. Are $|2, 1, -3\rangle$ and $|1, 1, 1\rangle$ perpendicular in $\mathbb{R}^3$ with the standard inner product?

18. Are $|7, 5, 3\rangle$ and $|1, -2, 1\rangle$ perpendicular in $\mathbb{R}^3$ with the standard inner product?

19. Find a vector perpendicular to $|2, -1, 2\rangle$ and $|1, -1, 2\rangle$ in $\mathbb{R}^3$ with the standard inner product.

20. Find the length of $|1, 2, 1\rangle$ in $\mathbb{R}^3$ with the standard inner product.

21. Find the length of $|3, -2, 5\rangle$ in $\mathbb{R}^3$ with the standard inner product.

22. Find the length of $|1, 0, -1\rangle$ in $\mathbb{R}^3$ with the standard inner product.

23. Find the angle between $|3, -2, 1\rangle$ and $|1, -1, 1\rangle$ in $\mathbb{R}^3$ with the standard inner product.

24. Find the angle between $|2, 1, -1\rangle$ and $|1, 0, 2\rangle$ in $\mathbb{R}^3$ with the standard inner product.

25. Determine (with proof) whether the function $\langle,\rangle\colon \mathfrak{R}^2 \times \mathfrak{R}^2 \to \mathbb{R}$ given by
$$\langle (\alpha_1\ \alpha_2), (\beta_1\ \beta_2)\rangle = \alpha_1\beta_1 - \alpha_2\beta_1 - \alpha_1\beta_2 + 2\alpha_2\beta_2$$
is an inner product on $\mathfrak{R}^2$.

26. Determine (with proof) whether the function $\langle,\rangle\colon \mathfrak{R}^2 \times \mathfrak{R}^2 \to \mathbb{R}$ given by
$$\langle (\alpha_1\ \alpha_2), (\beta_1\ \beta_2)\rangle = \alpha_1\beta_1 + \alpha_2\beta_1 + \alpha_1\beta_2 - \alpha_2\beta_2$$
is an inner product on $\mathfrak{R}^2$.

27. Determine (with proof) whether the function $\langle , \rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by
$$\langle (\alpha_1\ \alpha_2), (\beta_1\ \beta_2) \rangle = \alpha_1\beta_1 - \alpha_2\beta_1 + \alpha_1\beta_2 + 2\alpha_2\beta_2$$
is an inner product on $\mathbb{R}^2$.

28. Determine (with proof) whether the function $\langle , \rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by
$$\langle (\alpha_1\ \alpha_2), (\beta_1\ \beta_2) \rangle = \alpha_1^2\beta_1^2 + \alpha_2^2\beta_2^2$$
is an inner product on $\mathbb{R}^2$.

29. Determine (with proof) whether the function $\langle , \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by
$$\langle (\alpha_1\ \alpha_2, \alpha_3), (\beta_1\ \beta_2, \beta_3) \rangle = \alpha_1\beta_1 + 2\alpha_2\beta_2 + 3\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1\beta_3 + \alpha_3\beta_1 + 2\alpha_2\beta_3 + 2\alpha_3\beta_2$$
is an inner product on $\mathbb{R}^3$.

30. Determine (with proof) whether the function $\langle , \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by
$$\langle (\alpha_1\ \alpha_2, \alpha_3), (\beta_1\ \beta_2, \beta_3) \rangle = \alpha_1\beta_1 + 2\alpha_2\beta_2 + 3\alpha_3\beta_3 + 2\alpha_1\beta_2 + 2\alpha_1\beta_3 + 4\alpha_2\beta_3$$
is an inner product on $\mathbb{R}^3$.

31. Determine (with proof) whether the function $\langle , \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by
$$\langle (\alpha_1\ \alpha_2, \alpha_3), (\beta_1\ \beta_2, \beta_3) \rangle = \alpha_1\beta_1 + \alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 - \alpha_1\beta_3 - \alpha_3\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2$$
is an inner product on $\mathbb{R}^3$.

32. Determine (with proof) whether the function $\langle , \rangle \colon C(\mathbb{R}_{[-1,1]}, \mathbb{R}) \times C(\mathbb{R}_{[-1,1]}, \mathbb{R}) \to \mathbb{R}$ given by
$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$
is an inner product on $C(\mathbb{R}_{[-1,1]}, \mathbb{R})$.

33. Determine (with proof) whether the function $\langle , \rangle \colon C(\mathbb{R}_{[-1,1]}, \mathbb{R}) \times C(\mathbb{R}_{[-1,1]}, \mathbb{R}) \to \mathbb{R}$ given by
$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)(1-x^2)dx$$
is an inner product on $C(\mathbb{R}_{[-1,1]}, \mathbb{R})$.

34. Determine (with proof) whether the function $\langle , \rangle \colon C(\mathbb{R}_{[-1,1]}, \mathbb{R}) \times C(\mathbb{R}_{[-1,1]}, \mathbb{R}) \to \mathbb{R}$ given by
$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)x^2dx$$
is an inner product on $C(\mathbb{R}_{[-1,1]}, \mathbb{R})$.

35. Determine (with proof) whether the function $\langle , \rangle \colon M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ given by
$$\langle A, B \rangle = \mathrm{tr}(AB)$$
is an inner product on $M_{2\times 2}(\mathbb{R})$.

36. Determine (with proof) whether the function $\langle , \rangle \colon M_{2\times 2}(\mathbb{R}) \times M_{2\times 2}(\mathbb{R}) \to \mathbb{R}$ given by
$$\langle A, B \rangle = \det(AB)$$
is an inner product on $M_{2\times 2}(\mathbb{R})$.

8.2 Lengths, distances, and angles between vectors

37. Let $u = |1, 0, 2, -2\rangle$ and $v = |2, 1, -2, 0\rangle$ in $\mathbb{R}^4$ with the standard inner product. Compute $\|u\|$, $\|v\|$, $\langle u, v\rangle$ and the angle between u and v and verify that the Cauchy-Schwarz and the triangle inequalities hold.

38. Let $u = x$ and $v = \cos(\pi x)$ in $C(\mathbb{R}_{[0,1]}, \mathbb{R})$ with the standard inner product. Compute $\|u\|$, $\|v\|$, $\langle u, v\rangle$ and the angle between u and v and verify that the Cauchy-Schwarz and the triangle inequalities hold.

39. Let $u = (1, 0, 2)$ and $v = (2, 1, 2)$ in $\mathbb{R}^3$ with $\langle, \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle(\alpha_1\ \alpha_2, \alpha_3), (\beta_1\ \beta_2, \beta_3)\rangle = \alpha_1\beta_1 + 2\alpha_2\beta_2 + 3\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1\beta_3 + \alpha_3\beta_1 + 2\alpha_2\beta_3 + 2\alpha_3\beta_2.$$

Compute $\|u\|$, $\|v\|$, $\langle u, v\rangle$ and the angle between u and v and verify that the Cauchy-Schwarz and the triangle inequalities hold.

40. Let $u = (1, 0, 2)$ and $v = (2, 1, 2)$ in $\mathbb{R}^3$ with $\langle, \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle(\alpha_1\ \alpha_2, \alpha_3), (\beta_1\ \beta_2, \beta_3)\rangle = \alpha_1\beta_1 + 2\alpha_2\beta_2 + 3\alpha_3\beta_3 + 2\alpha_1\beta_2 + 2\alpha_1\beta_3 + 4\alpha_2\beta_3$$

Compute $\|u\|$, $\|v\|$, $\langle u, v\rangle$ and the angle between u and v and verify that the Cauchy-Schwarz and the triangle inequalities hold.

41. Let $u = (1, 0, 2)$ and $v = (2, 1, 2)$ in $\mathbb{R}^3$ with $\langle, \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle(\alpha_1\ \alpha_2, \alpha_3), (\beta_1\ \beta_2, \beta_3)\rangle = \alpha_1\beta_1 + \alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 - \alpha_1\beta_3 - \alpha_3\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2$$

Compute $\|u\|$, $\|v\|$, $\langle u, v\rangle$ and the angle between u and v and verify that the Cauchy-Schwarz and the triangle inequalities hold.

42. Let V be a inner product space and let $u, v \in V$. Show that

$$|\ \|u\| - \|v\|\ | \le \|u - v\|.$$

43. Let V be a inner product space and let $u, v \in V$. Show that

$$4\langle u, v\rangle = \|u + v\|^2 - \|u - v\|^2.$$

44. Let V be a inner product space and let $u, v \in V$. Show that

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

What does this say about the diagonals of a parallelogram?

45. Let V be a inner product space and let $u, v \in V$. Show that

$$d(u, v) \ge 0.$$

46. Let V be a inner product space and let $u, v \in V$. Show that

$$d(u, v) = d(v, u).$$

47. Let V be a inner product space and let $u, v \in V$. Show that

$$d(u, v) = 0 \quad \text{if and only if} \quad u = v.$$

8.3 Orthogonality and orthogonalization

48. Let V be an $\mathbb{F}$-vector space with an inner product $\langle , \rangle \colon V \times V \to \mathbb{F}$. Assume $B = (b_1, \ldots, b_n)$ is an ordered orthonormal basis of V and $x \in V$. Prove that

$$x = \langle x, b_1 \rangle b_1 + \cdots + \langle x, b_n \rangle b_n.$$

49. Let $\langle , \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be given by

$$\langle (u_1, u_2, u_3), (v_1, v_2, v_3) \rangle = u_1 v_1 + 2 u_2 v_2 + u_3 v_3.$$

(a) Prove that $S = \{(1,1,1), (1-1,1), (1,0-1)\}$ is an orthogonal sequence in $\mathbb{R}^3$ with respect to $\langle , \rangle$.

(b) Let

$$b_1 = \frac{1}{\|u\|} u, \qquad \text{where} \quad u = (1,1,1),$$
$$b_2 = \frac{1}{\|v\|} v, \qquad \text{where} \quad v = (1,-1,1),$$
$$b_3 = \frac{1}{\|w\|} w, \qquad \text{where} \quad w = (1,0,01).$$

Show that $\{b_1, b_2, b_3\}$ is an orthonormal sequence in $\mathbb{R}^3$ with respect to $\langle , \rangle$.

(c) Let $x = |1, 1, -1\rangle$ and $c_2, c_2, c_3 \in \mathbb{R}$ such that

$$x = c_1 b_1 + c_2 b_2 + c_3 b_3.$$

Compute c_1, c_2, c_3.

50. Let $V = \mathbb{R}^n$ and let $u \in V$ with $u \neq 0$. Let

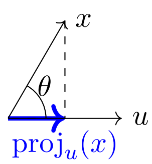
$$W = \mathbb{R}\text{-span}\{u\} = \{au \mid a \in \mathbb{R}\}.$$

(a) Show that W is a subspace of V.

(b) Find an orthonormal basis of W.

(c) Let $x \in V$. Prove that

$$\mathrm{proj}_W(x) = \mathrm{proj}_u(x).$$



51. Let $V = \mathbb{R}^3$ and let $W = \{|x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}$.

(a) Show that the set

$$\{b_1, b_2\} = \left\{ \tfrac{1}{\sqrt{2}} |1, -1, 0\rangle, \tfrac{1}{\sqrt{6}} |1, 1, -2\rangle \right\}$$

is an orthonormal basis of W with respect to the standard inner product on $\mathbb{R}^3$.

(b) Let $x = |1, 2, 3\rangle$. Compute $\mathrm{proj}_W(x)$.

(c) Compute the shortest distance from x to W.

52. Let $V = \mathbb{R}^3$ with the standard inner product. Let $S = \{v_1, v_2, v_3\}$ with

$$v_1 = |1,1,1\rangle, \quad v_2 = |0,1,1\rangle, \quad v_1 = |0,0,1\rangle.$$

Apply the Gram-Schmidt process to convert S into an orthonormal basis B.

53. Let V be a $\mathbb{C}$-vector space with inner product $\langle,\rangle\colon V \times V \to \mathbb{C}$.
Let $B = \{b_1, \ldots, b_k\}$ be an orthogonal set in V. Prove that B is linearly independent.

54. Show that $|2,3,-2,1,0\rangle$ and $|2,-1,1,0,2,1\rangle$ are orthogonal in $\mathbb{R}^4$ (with the standard symmetric inner product).

55. Show that $|i,1,-i\rangle$ and $|1-i,2,1+i\rangle$ are orthogonal in $\mathbb{C}^3$ (with the standard Hermitian inner product).

56. Show that 1 and $\cos(\pi x)$ are orthogonal in $C(\mathbb{R}_{[0,1]}, \mathbb{C})$ (with the standard Hermitian inner product).

57. Find all vectors which are orthogonal to $|-1,1,2,-1\rangle$ in $\mathbb{R}^4$ (with the standard symmetric inner product).

58. Find all vectors which are orthogonal to $|1-i,1+i\rangle$ in $\mathbb{C}^2$ (with the standard Hermitian inner product).

59. Assume that $m,n \in \mathbb{Z}$ and $m \neq n$. Show that $\cos(2m\pi x)$ and $\cos(2n\pi x)$ are orthogonal in in $C(\mathbb{R}_{[0,1]}, \mathbb{R})$ and find the length of $\cos(2m\pi x)$.

60. Find a quadratic polynomial orthogonal to 1 and x in $C(\mathbb{R}_{[0,1]}, \mathbb{R})$.

61. Let $a,b,c,d \in \mathbb{R}$. Find necessary and sufficient conditions for $a+bx$ and $c+dx$ to be orthogonal in $C(\mathbb{R}_{[0,1]}, \mathbb{R})$.

62. Show that $\sin(\pi x), \sin(2\pi x), \sin(3\pi x), , \ldots$ are orthogonal in $C(\mathbb{R}_{[0,1]}, \mathbb{R})$. Obtain an orthonormal set from these.

63. Use the Gram-Schmidt procedure to orthogonalize $\{|1,-1,1\rangle, |2,1,1\rangle, |1,0,1\rangle\}$ in $\mathbb{R}^3$.

64. Use the Gram-Schmidt procedure to orthogonalize $\{|1,-1,1,1\rangle, |0,1,0,1\rangle, |2,-1,1,1\rangle\}$ in $\mathbb{R}^4$.

65. Use the Gram-Schmidt procedure to orthogonalize $\{|1,-1,i\rangle, |i,1,2\rangle\}$ in $\mathbb{C}^3$.

66. Complete $\{|\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\rangle, |0,1,0\rangle\}$ to an orthonormal basis of $\mathbb{R}^3$.

67. Complete $\{\frac{1}{2}|1,i,1,i\rangle, \frac{1}{2}|i,1,i,1\rangle\}$ to an orthonormal basis of $\mathbb{C}^4$.

68. Find an orthonormal basis for $\mathbb{R}^3$ with $\langle,\rangle\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle(\alpha_1\ \alpha_2, \alpha_3), (\beta_1\ \beta_2, \beta_3)\rangle = \alpha_1\beta_1 + 2\alpha_2\beta_2 + 3\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_1\beta_3 + \alpha_3\beta_1 + 2\alpha_2\beta_3 + 2\alpha_3\beta_2.$$

69. Find an orthonormal basis for $\mathbb{R}^3$ with $\langle,\rangle\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle(\alpha_1\ \alpha_2, \alpha_3), (\beta_1\ \beta_2, \beta_3)\rangle = \alpha_1\beta_1 + 2\alpha_2\beta_2 + 3\alpha_3\beta_3 + 2\alpha_1\beta_2 + 2\alpha_1\beta_3 + 4\alpha_2\beta_3$$

70. Find an orthonormal basis for $\mathbb{H}^3$ with $\langle,\rangle\colon \mathbb{H}^3\times\mathbb{H}^3\to\mathbb{R}$ given by

$$\langle(\alpha_1\ \alpha_2,\alpha_3),(\beta_1\ \beta_2,\beta_3)\rangle = \alpha_1\beta_1+\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1-\alpha_1\beta_3-\alpha_3\beta_1+\alpha_2\beta_3+\alpha_3\beta_2$$

71. Apply Gram-Schmidt to $\{1,x,x^2,x^3\rangle$ to produce an orthonormal basis of $\mathbb{R}[x]_{\le 3}$.

72. Let $B=\{b_1,\ldots,b_n\}$ be an orthonormal basis of a real inner product space V. Prove that, if $v\in V$ then
$$v=\langle v,b_1\rangle b_1+\cdots\langle v,b_n\rangle b_n.$$

73. Specify an orthonormal basis of $\mathbb{C}^3$ and find the coordinates of $|1,i,-i\rangle$ with respect to this basis.

74. Specify an orthonormal basis of $\mathbb{R}[x]_{\le 2}$ and find the coordinates of x^2+1 and x^2-x+1 with respect to this basis.

75. Let $\mathbb{R}[x]_{\le 3}$ have inner product $\langle,\rangle\colon \mathbb{R}[x]_{\le 3}\times\mathbb{R}[x]_{\le 3}\to\mathbb{R}$ given by
$$\langle f,g\rangle=\int_{-1}^{1} f(x)g(x)(1-x^2)dx.$$
Apply Gram-Schmidt orthogonalization to $S=\{1,x,x^2,x^3\}$ to obtain an orthonormal basis of $\mathbb{R}[x]_{\le 3}$.

76. Let V be a real inner product space and let $B=\{b_1,\ldots,b_n\}$ be an orthonormal basis of V. Let $T\colon V\to V$ be a linear transformation. Let $i,j\in\{1,\ldots,n\}$. Prove that the (i,j) entry of the matrix of T with respect to the basis B is $\langle Tv_j,v_i\rangle$.

77. Let V be a real inner product space and let $u,v\in V$ such that $\|u\|=\|v\|$. Prove that
$$u+v\quad\text{is orthogonal to}\quad u-v.$$
What does this say about the diagonals of a rhombus?

78. Let V be a inner product space and let $u,v\in V$. Show that
$$\|u+v\|^2+\|u-v\|^2=2(\|u\|^2+\|v\|^2).$$
What does this say about the diagonals of a parallelogram in $\mathbb{R}^2$ (or for that matter, any dimension)?

9 Determinants and characteristic and minimal polynomials

9.1 Determinants

1. Let $A = \begin{pmatrix} 2 & 6 & 9 \\ 0 & 3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$. Find the determinant of A by computing the normal form of A.

2. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the determinant of A by computing the normal form of A.

3. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Compute $\mathrm{tr}(A)$.

4. Let $A, B \in M_{n\times n}(\mathbb{Q})$ and let $c \in \mathbb{Q}$. Prove that
$$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B), \qquad \mathrm{tr}(cA) = c\,\mathrm{tr}(A) \qquad \text{and} \qquad \mathrm{tr}(AB) = \mathrm{tr}(BA).$$

5. Let $n \in \mathbb{Z}_{>0}$. Let $A \in M_{n\times n}(\mathbb{Q})$ and let $P \in GL_n(\mathbb{Q})$ so that P is an invertible $n \times n$ matrix. Prove that
$$\det(PAP^{-1}) = \det(A) \qquad \text{and} \qquad \mathrm{tr}(PAP^{-1}) = \mathrm{tr}(A).$$

6. Let $A \in M_{n\times n}(\mathbb{Q})$. Prove that
$$\frac{1}{\det(A)} = \det(A^{-1}).$$

7. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.
   (a) Compute the $(2,3)$ cofactor of A.
   (b) Use cofactor expansion along the third row to compute $\det(A)$.

8. Using cofactor expansion down the fourth column, compute the determinant of
$$A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 4 \end{pmatrix}.$$

9. Find $\det \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$.

10. Find $\det \begin{pmatrix} 2 & 0 & -1 \\ 1 & 2 & 2 \\ 3 & 2 & 4 \end{pmatrix}$.

11. Find $\det \begin{pmatrix} 2 & 1 & 2 \\ -1 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$.

12. Find $\det\begin{pmatrix} -2 & 1 & 0 \\ 5 & 0 & 1 \\ 0 & 2 & 2 \end{pmatrix}$.

13. Find $\det\begin{pmatrix} 2 & 3 & 1 \\ 1 & 0 & 1 \\ 3 & 3 & 2 \end{pmatrix}$.

14. Factor $\det\begin{pmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{pmatrix}$.

15. Factor $\det\begin{pmatrix} a^2 & (b+c)^2 & bc \\ b^2 & (c+a)^2 & ca \\ c^2 & (a+b)^2 & ab \end{pmatrix}$.

16. Factor $\det\begin{pmatrix} b^2+c^2 & (b-c)^2 & a^2+2bc \\ c^2+a^2 & (c-1)^2 & b^2+2ca \\ a^2+b^2 & (a-b)^2 & c^2+2a \end{pmatrix}$.

17. Factor $\det\begin{pmatrix} (x-a)^2 & (x+a)^2 & 2y^2+2z^2 \\ (y-a)^2 & (y+a)^2 & 2z^2+2x^2 \\ (z-a)^2 & (z+a)^2 & 2x^2+2y^2 \end{pmatrix}$.

18. Factor $\det\begin{pmatrix} a(b+c) & (b+c)^2 & a^2-2bc \\ b(c+a) & (c+a)^2 & b^2-2ca \\ c(a+b) & (a+b)^2 & c^2-2ab \end{pmatrix}$.

19. Determine which $\theta\in\mathbb{R}$ satisfy $\det\begin{pmatrix} 1+\sin^2(\theta) & \cos^2\theta & 4\sin(2\theta) \\ \sin^2(\theta) & 1+\cos^2(\theta) & 4\sin(2\theta) \\ \sin^2(\theta) & \cos^2(\theta) & 1+4\sin(2\theta) \end{pmatrix}=0$.

20. Let $a,b,c\in\mathbb{C}$. Determine which $x\in\mathbb{C}$ satisfy $\det\begin{pmatrix} a-x & b-x & c \\ a-x & c & b-x \\ a & b-x & c-x \end{pmatrix}=0$.

21. Compute $\det\begin{pmatrix} 0 & -1 & 2 & 1 \\ -4 & 3 & -3 & 5 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 1 \end{pmatrix}$.

22. Compute $\det\begin{pmatrix} 2 & 1 & 0 & 1 \\ -1 & 2 & 1 & 0 \\ 1 & 2 & 5 & 3 \\ 0 & -1 & 1 & 1 \end{pmatrix}$.

23. Compute $\det\begin{pmatrix} 2 & -1 & 1 & 1 \\ 5 & 6 & 1 & 2 \\ 1 & -2 & 1 & 1 \\ 3 & -2 & 1 & 1 \end{pmatrix}$.

24. Let $a, b, c \in \mathbb{C}$. Find the values $x \in \mathbb{C}$ such that $\det \begin{pmatrix} x & a & b & c \\ a & x & b & c \\ a & b & x & c \\ a & b & c & x \end{pmatrix} = 0$.

25. Let $a, b, c \in \mathbb{C}$ and let

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a+b & a+c \\ 1 & b+a & 0 & b+c \\ 1 & c+a & c+b & 0 \end{pmatrix}.$$

   (a) Show that $\det(A) = -4(ab + bc + ca)$.

   (b) Assume a, b, c are distinct cube roots of unity. Find $\det(A)$.

   (c) Assume a, b, c are distinct fourth roots of unity. Find $\det(A)$.

26. Show that $\det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & -1 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & 1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 1 \end{pmatrix} = 2^{n-1}$.

27. Show that $\det \begin{pmatrix} x & 1 & 1 & \cdots & 1 & 1 \\ 1 & x & 0 & \cdots & 0 & 0 \\ 1 & 1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 1 & 1 & 1 & \cdots & 1 & x \end{pmatrix} = (x-1)((x-1)^{n-2} + x^{n-1})$.

28. Show that $\det \begin{pmatrix} 1+x^2 & x & 0 & \cdots & 0 & 0 \\ x & 1+x^2 & x & \cdots & 0 & 0 \\ 0 & x & 1+x^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & x & 1+x^2 \end{pmatrix} = 1 + x^2 + \cdots + x^{2n}$.

29. Show that $\det \begin{pmatrix} a+b & a & a & \cdots & a \\ a & a+b & a & \cdots & a \\ \vdots & \vdots & \vdots & & \vdots \\ a & a & a & \cdots & a+b \end{pmatrix} = b^{n-1}(na + b)$.

30. Let $A \in M_{n\times n}(\mathbb{C})$ with

$$A(1,1) \neq 0 \qquad \text{and} \qquad A(i,j) \neq 0 \text{ when } i + j > n+1$$

and $A(i,j) = 0$ otherwise. Find $\det(A)$.

31. Let $A_n \in M_{n\times n}(\mathbb{Q})$ be given by

$$A_n = \begin{pmatrix} -2 & 4 & 0 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 4 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -2 & 4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 \end{pmatrix}.$$

(a) Show that if $n = 3k-1$ then $\det(A_n) = 0$.

(b) Assume $n = 3k$ and find $\det(A_n)$.

32. Let $D\colon M_n(\mathbb{R}) \to \mathbb{R}$ be such that

$$\text{if } A, B \in M_n(\mathbb{R}) \quad \text{then} \quad D(A)D(B) = D(AB).$$

(a) Assume $D(1) = 1$. Show that if A is invertible then $D(A) \neq 0$.

(b) Assume $D(1) = 0$. Show that if $A \in M_n(\mathbb{C})$ then $D(A) = 0$.

33. Let $D\colon M_n(\mathbb{R}) \to \mathbb{R}$ be such that

$$\text{if } A, B \in M_n(\mathbb{R}) \quad \text{then} \quad D(A)D(B) = D(AB).$$

Assume that $A \in M_n(\mathbb{C})$ has a row of zeros. Show that $D(A) = 0$.

34. Let $A_n \in M_n(\mathbb{C})$ be given by $A(i,j) = |i-j|$. Show that $\det(A_n) = (-1)^{n-1}(n-1)2^{n-2}$.

35. Determine if $\begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 4 & 5 & 2 \end{pmatrix}$ is invertible.

36. Determine if $\begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 3 & 5 \end{pmatrix}$ is invertible.

37. Determine if $\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}$ is invertible.

38. Let $\alpha, \beta, \gamma \in \mathbb{C}$. For $r \in \mathbb{Z}_{>0}$ let

$$s_r = \alpha^r + \beta^r + \gamma^r.$$

(a) Express the determinant $\det \begin{pmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix}$ as a product of two determinants.

(b) Show that $\det \begin{pmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{pmatrix} = (\alpha - \beta^2)(\alpha - \gamma)^2(\beta - \gamma)^2$.

(c) Evaluate $\det \begin{pmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_4 \\ s_2 & s_3 & s_5 \end{pmatrix}$.

39. Let A be the matrix
$$\begin{pmatrix} a+bx+cx^2 & a+by+cy^2 & a+bz+cz^2 \\ c+ax+bx^2 & c+ay+by^2 & c+ax+bx^2 \\ b+cx+ax^2 & b+cy+ay^2 & b+cx+ax^2 \end{pmatrix}$$
and let B be the matrix
$$\begin{pmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}.$$
Find a matrix C such that $A = CB$. Show that the determinant of A is
$$\det \begin{pmatrix} a+b+c & a+b+c & a+b+c \\ ax+cy+bz & bx+ay+cz & cx+by+az \\ ax^2+cy^2+bz^2 & bx^2+ay^2+cz^2 & cx^2+by^2+az^2 \end{pmatrix}.$$

40. Use Cramer's method to solve the following system of linear equations.
$$\begin{aligned} x-y+2z &= 1, \\ 2x+y+z &= 2, \\ x-3y+z &= 1. \end{aligned}$$

41. Use Cramer's method to solve the following system of linear equations.
$$\begin{aligned} 2x+3y-5z &= 4, \\ x+7y-2z &= 1, \\ 5x-11y+2z &= -2. \end{aligned}$$

42. For what values of t is $A = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & -1 \\ t & 0 & 1 \end{pmatrix}$ invertible? Find the inverse for each value of t such that A is invertible.

43. Let $A \in M_n(\mathbb{Q})$ and let $(\text{adj } A)$ be the adjugate of A. Prove that $\det(\text{adj } A) = (\det(A))^{n-1}$.

44. Let $x \in \mathbb{C}$ and let
$$A = \begin{pmatrix} x+1 & 0 & -1 \\ 0 & x+1 & -2 \\ 1 & 1 & x-2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & x-1 \end{pmatrix}.$$
   (a) Find the adjugate of A.
   (b) For which $x \in \mathbb{C}$ is $(\text{adj } A)B$ invertible?

9.2 Characteristic polynomials and minimal polynomials

45. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. Find the characteristic polynomial of
$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \alpha_1 \\ 1 & 0 & \cdots & 0 & \alpha_2 \\ 0 & 1 & \cdots & 0 & \alpha_3 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \alpha_n \end{pmatrix}$$

46. Let $a, b \in \mathbb{R}$. Find the characteristic polynomial of
$$\begin{pmatrix} 1+b & a & a^2 & \cdots & a^{n-1} \\ 1 & a+b & a^2 & \cdots & \alpha_2 \\ 1 & a & a^2+b & \cdots & \alpha_3 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & a & a^2 & \cdots & a^{n-1}+b \end{pmatrix}$$

47. Find the minimal polynomial of $A. = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

48. Find the minimal polynomial of $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

49. Find the minimal polynomial of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

50. Find the minimal polynomial of $A = \begin{pmatrix} 1 & -4 & 2 & 4 \\ 2 & -1 & -1 & 2 \\ 0 & -4 & 3 & 4 \\ 2 & 0 & -2 & 1 \end{pmatrix}$.

51. Find the minimal polynomial and the characteristic polynomial of the matrix
$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}.$$

52. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n$. Find an $n \times n$ matrix with $f(x)$ as the minimal polynomial.

53. Assume that A is a matrix with characteristic polynomial $(x-1)^3(x+2)^2(x-3)$.
    (a) Find all possible minimal polynomials of A.
    (b) For each minimal polynomial given in A find a matrix A that has that minimal polynomial.

54. Give a direct proof of the Cayley-Hamilton theorem for diagonal matrices.

55. Give a direct proof of the Cayley-Hamilton theorem for triangular matrices.

56. Calculate the minimal polynomial of $A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{pmatrix}$. Determine if A is diagonalizable.

57. Calculate the minimal polynomial of $A = \begin{pmatrix} 2 & 0 & -1 \\ -1 & 2 & 2 \\ 1 & -1 & -1 \end{pmatrix}$. Determine if A is diagonalizable.

58. Calculate the minimal polynomial of $A = \begin{pmatrix} 1 & 4 & -2 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 4 & -1 & -4 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Determine if A is diagonalizable.

59. Calculate the minimal polynomial of $A = \begin{pmatrix} 3 & -4 & 0 & 5 \\ 1 & -1 & 0 & 1 \\ 2 & -3 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Determine if A is diagonalizable.

9.3 Conjugacy invariants

60. (The Cauchy-Binet theorem) Let $n^{\wedge k}$ be the set of subsets of $\{1, \ldots, n\}$ with k elements,

$$n^{\wedge k} = \{I \subseteq \{1, \ldots, n\} \mid |I| = k\}.$$

Define a matrix $A^{\wedge k} \in M_{n^{\wedge k}}(\mathbb{F})$ by

$$A^{\wedge k}(I, J) = \det(A_{IJ}),$$

where A_{IJ} is the $k \times k$ matrix consisting of the rows of A specified by I and the columns specified by J.

Let $I, J \subseteq \{1, \ldots, n\}$ with $|I| = |J| = k$. Define

$$(A^{\wedge k})_{IJ} = \sum_{w \in S_k} \det(w) A(i_1, j_{w(1)}) \cdots A(i_k, j_{w(k)}).$$

Then

$$((AB)^{\wedge k})_{IJ} = \sum_{K \in \mathbb{Z}^{\wedge k}_{[1,n]}} (A^{\wedge k})_{IK} (B^{\wedge k})_{KJ}.$$

In other words, if $A, B \in M_n(\mathbb{F})$ then

$$(AB)^{\wedge k} = A^{\wedge k} B^{\wedge k}.$$

10 Geometry, other applications and learning to do proofs

10.1 Geometry in $\mathbb{R}^n$

1. Let
$$e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$
Compute $3e_1 + 5e_2 + (-2)e_3 + 0e_4$.

2. Let $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that
$$\langle y, x\rangle = \langle x, y\rangle, \qquad \langle x, y\rangle = x^T y, \qquad \|x\| = \sqrt{\langle x, x\rangle}, \qquad d(x,y) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}.$$

3. Let $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that
$$\langle y, x\rangle = \langle x, y\rangle, \qquad \langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle, \qquad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$
$$\langle x, cy\rangle = c\langle x, y\rangle, \qquad \langle cx, y\rangle = c\langle x, y\rangle, \qquad \|cx\| = |c| \cdot \|x\|.$$

4. (Cauchy-Schwarz) Let $u, v \in \mathbb{R}^n$. Prove that
$$|\langle u, v\rangle| \le \|u\| \cdot \|v\|.$$

5. (Triangle inequality) Let $u, v \in \mathbb{R}^n$. Prove that
$$\|u + v\| \le \|u\| + \|v\|$$

6. Let $u = |1, 3, 1, 2\rangle$ and $v = |2, 1, -1, 3\rangle$ in $\mathbb{R}^4$. Compute $u - v$ and $d(u, v)$.

7. Let $u = |0, 2, 2, -1\rangle$ and $v = |-1, 1, 1, -1\rangle$ in $\mathbb{R}^4$. Compute $\langle u, v\rangle$, $\|u\|$ and $\|v\|$ and check that $|\langle u, v\rangle| \le \|u\| \cdot \|v\|$.

8. Let $u = (2, -1, -2)$ and $v = (2, 1, 3)$. Find vectors v_1 and v_2 such that
$$v = v_1 + v_2$$
where v_1 is parallel to u and v_2 is perpendicular to u.

9. Determine the vector, parametric and Cartesian equations of the line through the points $P = (-1, 2, 3)$ and $Q = (4, -2, 5)$.

10. Find a vector equation of the 'friendly' line through the point $(2, 0, 1)$ that is parallel to the 'enemy' line
$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-6}{2}.$$
Does the point $(0, 4, -3)$ lie on the 'friendly' line?

11. Find the vector equation for the plane in $\mathbb{R}^3$ containing the points $P = |1, 0, 2\rangle$ and $Q = |1, 2, 3\rangle$ and $R = |4, 5, 6\rangle$.

12. Where does the line
$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$$
intersect the plane $3x+2y+z=20$?

13. Find a vector form for the line of intersection of the two planes $x+3y+2z=6$ and $3x+2y+z=11$.

14. Let $\hat{\imath},\hat{\jmath},\hat{k}\in\mathbb{R}^3$ be given by
$$\hat{\imath}=|1,0,0\rangle,\qquad \hat{\jmath}=|0,1,0\rangle,\qquad \hat{k}=|0,0,1\rangle.$$
Compute $5\hat{\imath}+(-1)\hat{\jmath}+(-4)\hat{k}$.

15. Let $u=|u_1,u_2,u_3\rangle, v=|v_1,v_2,v_3\rangle, w=|w_1,w_2,w_3\rangle$ In $\mathbb{R}^3$. Prove that
$$u\times v=\det\begin{pmatrix}u_2&u_3\\v_2&v_3\end{pmatrix}\hat{\imath}-\det\begin{pmatrix}u_1&u_3\\v_1&v_3\end{pmatrix}\hat{\jmath}+\det\begin{pmatrix}u_1&u_2\\v_1&v_2\end{pmatrix}\hat{k},\qquad\text{and}$$
$$\langle u,v\times w\rangle=\det\begin{pmatrix}u_1&u_2&u_3\\v_1&v_2&v_3\\w_1&w_2&w_3\end{pmatrix}.$$

16. Let $u=|u_1,u_2,u_3\rangle, v=|v_1,v_2,v_3\rangle, w=|w_1,w_2,w_3\rangle$ In $\mathbb{R}^3$. Prove that
$$\langle v,v\times w\rangle=0\qquad\text{and}\qquad\langle w,v\times w\rangle=0.$$
Conclude that $v\times w$ is perpendicular to both v and w.

17. Find a vector perpendicular to both $|1,1,1\rangle$ and $|1,-1,-2\rangle$.

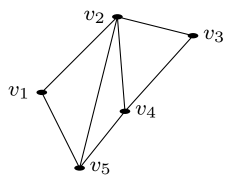
18. Find the volume of the parallelipiped with adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$, where
$$P=|2,0,-1\rangle,\qquad Q=|4,1,0\rangle,\qquad R=|3,-1,1\rangle\qquad\text{and}\qquad S=|2,-2,2\rangle.$$

19. Find the area of the triangle in $\mathbb{R}^3$ with vertices $|2,-5,4\rangle$, $|3,-4,5\rangle$ and $|3,-6,2\rangle$.

20. Find the Cartesian equation of the plane with vector form
$$|x,y,z\rangle=s\,|1,-1,0\rangle+t\,|2,0,1\rangle+|-1,1,1\rangle,\ \text{with } s,t\in\mathbb{R}.$$

10.2 Some further applications

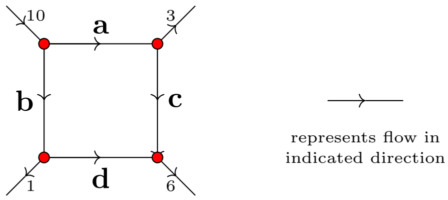
21. Use adjacency matrices to compute the number of paths of length 3 from vertex i to vertex j in the following graph for every pair (i,j) with $i,j\in\{1,\dots,5\}$.



22. At each node • require

$$(\text{sum of flows in}) = (\text{sum of flows out}).$$

Determine a, b, c and d in the network



23. (Ciphers by matrices) Before departure agree with your guardian angel on the encryption scheme

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 3 & 2 \end{pmatrix}.$$

Convert the letters A, . . . , Z to numbers $1, \ldots, 26$.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	$\cdots$
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	$\cdots$

The message

SENDMONEY becomes $19, 5, 14, 4, 13, 15, 14, 5, 25$.

Putting this message in a 3×3 matrix B and multiplying by the ecryption matrix produces the encrpyted message.

$$B = \begin{pmatrix} 19 & 4 & 14 \\ 5 & 13 & 5 \\ 14 & 15 & 25 \end{pmatrix} \qquad \text{and} \qquad AB = \begin{pmatrix} 43 & 45 & 49 \\ 105 & 118 & 128 \\ 81 & 77 & 93 \end{pmatrix}$$

so that the encrypted message is

$$43, 105, 81, 45, 118, 77, 49, 128, 93.$$

To decrypt the message multiply by A^{-1}.

$$A^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ -4 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad A^{-1} \begin{pmatrix} 43 & 45 & 49 \\ 105 & 118 & 128 \\ 81 & 77 & 93 \end{pmatrix} = \begin{pmatrix} 19 & 4 & 14 \\ 5 & 13 & 5 \\ 14 & 15 & 25 \end{pmatrix}.$$

giving $19, 5, 14, 4, 13, 15, 14, 5, 25$. In other words, SENDMONEY. Now! Please! ASAP!

$$40, 100, 61, 70, 167, 113$$

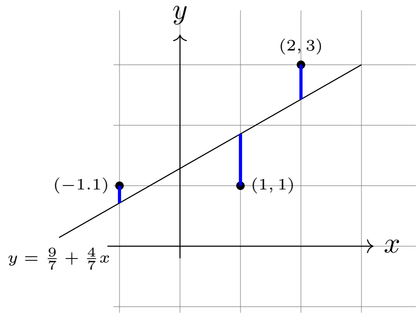
24. Suppose that the data set of assignment and exam marks for 7 students is

Student	*AssignmentMark*	*ExamMark*
$S1$	99	100
$S2$	80	82.5
$S3$	79	79
$S4$	75.5	82.5
$S5$	87.5	91
$S6$	67	67.5
$S7$	76	68

Calculate the mean assignment mark μ_A, the mean exam mark μ_E and the and the correlation $\mathrm{corr}(A, E) = \cos(\theta(A - \mu_A, E - \mu_E))$ between the assignment marks and the exam marks.

25. Given the data set $D = \{(-1, 1), (1, 1), (2, 3)\}$ let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$



Compute

$$\vec{s} = (A^t A)^{-1} A^t \vec{y} = \begin{pmatrix} a_{\text{best}} \\ b_{\text{best}} \end{pmatrix}$$

and determine the list of best fit $y = a_{\text{best}} + b_{\text{best}} x$ for the data set D.

10.3 Learning to do proofs

26. Prove the following theorem.

Theorem 10.1. ***(Exchange Theorem)*** *Let V be an $\mathbb{F}$-vector space. Let $B = \{v_1, \ldots, v_k\}$ be a basis of V and let $D = \{d_1, \ldots, d_\ell\}$ be another basis of V. Then there exists $d_{i_1} \in D$ such that*

$$\{d_{i_1}, b_2, b_3, \ldots, b_k\} \quad \textit{is a basis of } V.$$

27. Prove the following theorem.

Theorem 10.2. ***(Basis Minimax Theorem)*** *Let V be an $\mathbb{F}$-vector space and let B be a subset of V. The following are equivalent:*

(a) B is a basis of V.

(b) B is a minimal spanning set of V.

(c) B is a maximal linearly independent set of V.

28. Prove the following theorem.

Theorem 10.3. ***(Dimension Theorem)*** *Let V be an $\mathbb{F}$-vector space. Any two bases of V have the same number of elements.*

29. Let $W = \{|t, t, t\rangle \mid t \in \mathbb{R}\}$, a subset of $\mathbb{R}^3$.

(a) Show that W is a subspace of $\mathbb{R}^3$.

(b) Let S be the subset of $\mathbb{R}^3$ given by $S = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}$. Show that S is a spanning set for W.

(c) Show that S is not a minimal spanning set for W.

(d) Remove some elements of S to produce a minimal spanning set for W.

30. Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$.

    (a) Prove that If $\ker(A) = 0$ then the columns of A are linearly independent.

    (b) Prove that $\mathrm{im}(A) = \mathbb{R}\text{-span}\{\text{columns of } A\}$.

31. Prove the following theorem.

    Theorem 10.4. ***(Invertible matrices are square)*** *Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$. Suppose there exists*
    $$P \in M_{s\times t}(\mathbb{R}) \quad \textit{be such that} \quad PA = 1.$$
    Suppose there exists
    $$Q \in M_{s\times t}(\mathbb{R}) \quad \textit{be such that} \quad AQ = 1.$$
    Then

    (a) $\ker(A) = 0$.

    (b) $im(A) = \mathbb{R}^t$.

    (c) The set of columns of A is a basis of $\mathbb{Q}^t$.

    (d) $s = t$.

    (e) $P = Q$.

32. **PF4** Prove the following proposition.

    Proposition 10.5. ***(Span is a subspace)*** *Let V be a $\mathbb{R}$-vector space. Let $B = \{b_1, \ldots, b_k\}$ be a subset of V. Then $\mathbb{R}$-span(B) is a subspace of V.*

33. Prove the following proposition.

    Proposition 10.6. *Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$.*
    If $\ker(A) = 0$ *then the columns of A are linearly independent.*

34. Prove the following proposition.

    Proposition 10.7. *Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$. Then*
    $$im(A) = \mathbb{R}\text{-}span\{columns\ of\ A\}.$$

35. Prove the following proposition.

    Proposition 10.8. *Let $A \in M_{t\times s}(\mathbb{Q})$. Then* $\ker(A)$ *is a subspace of $\mathbb{Q}^s$.*

36. Prove the following proposition.

Proposition 10.9. *Let $A \in M_{t\times s}(\mathbb{Q})$. Then im$(A)$ is a subspace of $\mathbb{Q}^t$.*

37. Let $T\colon V \to W$ be an $\mathbb{R}$-linear transformation.
    Show that $\ker(T) = \{v \in V \mid T(v) = 0\}$ is a subspace of V.

38. Let $T\colon V \to W$ be an $\mathbb{R}$-linear transformation.
    Show that $\mathrm{im}(T) = \{T(v) \mid v \in V\}$ is a subspace of W

39. Prove the following proposition.

    Proposition 10.10. *Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$. Let $a_1, \ldots, a_s$ be the columns of A. Then*
    $$im(A) = span\{a_1, \ldots, a_s\}.$$