5 Vector spaces and linear transformations

5.1 Subspaces

VS1HS84.

(a) Is the line $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x + 1\}$ a subspace of $\mathbb{R}^2$?

(b) Is the line $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x\}$ a subspace of $\mathbb{R}^2$?

Solution. (a) Let $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x + 1\}$.
Since $0 = |0, 0\rangle$ and $0 \neq 2 \cdot 0 + 1$ then $0 \notin L$.
So L is not a subspace of $\mathbb{R}^2$.
(b) Let $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x\}$.
(ba) Since $|0, 0\rangle = |0, 2 \cdot 0\rangle$ then $|0, 0\rangle \in L$.
(bb) Assume $|a, 2a\rangle, |b, 2b\rangle \in L$. Then

$$|a, 2a\rangle + |b, 2b\rangle = |(a + b), 2(a + b)\rangle \in L.$$

(bc) Assume $|a, 2a\rangle \in L$ and $c \in \mathbb{R}$. Then

$$c \cdot |a, 2a\rangle = |(ca), 2(ca)\rangle \in L.$$

So L is a subspace of $\mathbb{R}^2$. $\square$

VS2HS94. Let $T : V \to W$ be an $\mathbb{R}$-linear transformation.
Show that $\ker(T) = \{v \in V \mid T(v) = 0\}$ is a subspace of V.

Solution. Let $v_1, v_2 \in \ker(T)$. Then

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0 + 0 = 0. \quad \text{So } v_1 + v_2 \in \ker(T).$$

Subtracting $T(0)$ from each side of the equation $T(0) = T(0 + 0) = T(0) + T(0)$ gives

$$0 = T(0), \quad \text{and so} \quad 0 \in \ker(T).$$

Let $v \in \ker(T)$ and let $c \in \mathbb{R}$. Then

$$T(cv) = cT(v) = c \cdot 0 = 0 \quad \text{and so} \quad cv \in \ker(T).$$

So $\ker(T)$ is a subspace of V. $\square$

VS3HS95. Let $T : V \to W$ be an $\mathbb{R}$-linear transformation.
Show that $\mathrm{im}(T) = \{T(v) \mid v \in V\}$ is a subspace of W.

Solution. Subtracting $T(0)$ from each side of the equation $T(0) = T(0 + 0) = T(0) + T(0)$ gives

$$0 = T(0), \quad \text{and so} \quad 0 \in \mathrm{im}(T).$$

Let $w_1, w_2 \in W$. Then there exist $v_1, v_2 \in V$ such that

$$T(v_1) = w_1 \quad \text{and} \quad T(v_2) = w_2.$$

Then $w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2)$,

$$\text{and so} \quad w_1 + w_2 \in \mathrm{im}(T).$$

Let $w \in W$ and let $c \in \mathbb{R}$. Then there exists $v \in V$ such that

$$T(v) = w.$$

Then $cw = cT(v) = T(cv)$

$$\text{and so} \quad cw \in \mathrm{im}(T).$$

So $\mathrm{im}(T)$ is a subspace of W. □

VS4HS140. Is $W = \{|x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}$ a $\mathbb{R}$-subspace of $\mathbb{R}^3$?

Solution.

A $\mathbb{R}$-*subspace of* $\mathbb{R}^3$ is a subset $W \subseteq \mathbb{R}^3$ such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,
(b) $0 \in W$,
(c) If $w \in W$ then $-w \in W$,
(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

To show: W is a subspace of $\mathbb{R}^3$.

(a) Assume $w_1 = |a, b, c\rangle \in W$ and $w_2 = |x, y, z\rangle \in W$.
Then $a + b + c = 0$ and $x + y + z = 0$.
Then $w_1+w_2 = |a+x, b+y, c+z\rangle$ and $(a+x)+(b+y)+(c+z) = (a+b+c)+(x+y+z) = 0+0 = 0$.
So $w_1 + w_2 \in W$.
(b) $0 = |0, 0, 0\rangle$ satisfies $0 + 0 + 0 = 0$. So $0 \in W$.
(c) Assume $w = |x, y, z\rangle \in W$.
Then $x + y + z = 0$.
Then $-w = |-x, -y, -z\rangle$ and $(-x) + (-y) + (-z) = -(x + y + z) = -0 = 0$.
So $-w \in W$.
(d) Assume $w = |x, y, z\rangle \in W$ and $c \in \mathbb{R}$.
Then $x + y + z = 0$.
Then $cw = |cx, cy, cz\rangle$ and $cx + cy + cz = c(x + y + z) = c \cdot 0 = 0$.
So $cw \in W$.

So W is a subspace of $\mathbb{R}^3$. □

VS5HS141. Is $W = \{a_1x + a_2x^2 \mid a_1, a_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}[x]_{\le 2}$?

Solution.

A *subspace of* $\mathbb{R}[x]_{\le 2}$ is a subset $W \subseteq \mathbb{R}[x]_{\le 2}$ such that

(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,
(b) $0 \in W$,
(c) If $w \in W$ then $-w \in W$,
(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.

To show: W is a subspace of $\mathbb{R}[x]_{\le 2}$.

(a) Assume $w_1 = a_1x + a_2x^2 \in W$ and $w_2 = b_1x + b_2x^2 \in W$.

Then $a_1, a_2 \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.
Then $w_1+w_2 = a_1x+a_2x^2+b_1x+b_2x^2 = (a_1+a_1)x+(b_1+b_2)x^2$ and $a_1+b_1 \in \mathbb{R}$ and $a_2+b_2 \in \mathbb{R}$.
So $w_1 + w_2 \in W$.
(b) $0 = 0x + 0x^2$ satisfies $0 \in W$. So $0 \in W$.
(c) Assume $w = a_1x + a_2x^2 \in W$.
Then $a_1, a_2 \in \mathbb{R}$.
Then $-w = -(a_1x + a_2x^2) = -a_1x + (-a_2)x^2$ and $-a_1 \in \mathbb{R}$ and $-a_2 \in \mathbb{R}$.
So $-w \in W$.
(d) Assume $w = a_1x + a_2x^2 \in W$ and $c \in \mathbb{R}$.
Then $a_1, a_2 \in \mathbb{R}$.
Then $cw = c(a_1x + a_2x^2) = (ca_1)x + (ca_2)x^2$ and $ca_1 \in \mathbb{R}$ and $ca_2 \in \mathbb{R}$.
So $cw \in W$.
So W is a subspace of $\mathbb{R}[x]_{\le 2}$. □

VS6HS142. Is the set of real 2×2 matrices whose trace is equal to 0 a subspace of $M_{2\times 2}(\mathbb{R})$?

Solution.

A *subspace of* $M_{2\times 2}(\mathbb{R})$ is a subset $W \subseteq M_{2\times 2}(\mathbb{R})$ such that
(a) If $w_1, w_2 \in W$ then $w_1 + w_2 \in W$,
(b) $0 \in W$,
(c) If $w \in W$ then $-w \in W$,
(d) If $w \in W$ and $c \in \mathbb{R}$ then $cw \in W$.
The set of real 2×2 matrices whose trace is equal to 0 is

$$W = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \;\Big|\; a_{11} + a_{22} = 0 \right\}.$$

To show: W is a subspace of $M_{2\times 2}(\mathbb{R})$
(a) Assume $w_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in W$ and $w_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \in W$.
Then $a_{11} + a_{22} = 0$ and $b_{11} + b_{22} = 0$.
Then $w_1 + w_2 = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$ and

$$(a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22}) = 0 + 0 = 0.$$

So $w_1 + w_2 \in W$.
(b) $0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $(0, 0, 0)$ satisfies $0 + 0 = 0$. So $0 \in W$.
(c) Assume $w = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in W$.
Then $a_{11} + a_{22} = 0$.
Then $-w = -\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$ and $(-a_{11}) + (-a_{22}) = -(a_{11} + a_{22}) = -0 = 0$.
So $-w \in W$.
(d) Assume $w = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in W$ and $c \in \mathbb{R}$.

Then $a_{11} + a_{22} = 0$.

Then $cw = c\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{pmatrix}$ and

$$ca_{11} + ca_{22} = c(a_{11} + a_{22}) = c \cdot 0 = 0.$$

So $cw \in W$.

So W is a subspace of $M_{2\times 2}(\mathbb{R})$. □

VS7HS143. Is

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; ad - bc = 0 \right\} \text{ a subspace of } M_2(\mathbb{R})?.$$

Solution.

A *subspace of* $M_{2\times 2}(\mathbb{R})$ is a subset $S \subseteq M_{2\times 2}(\mathbb{R})$ such that

(a) If $w_1, w_2 \in S$ then $w_1 + w_2 \in S$,

(b) $0 \in S$,

(c) If $w \in S$ then $-w \in S$,

(d) If $w \in S$ and $c \in \mathbb{R}$ then $cw \in S$.

To show: S is a subspace of $M_2(\mathbb{R})$.

Let $w_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since $1 \cdot 0 - 0 \cdot 0 = 0 - 0 = 0$ then $w_1 \in S$.

Let $w_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since $0 \cdot 1 - 0 \cdot 0 = 0 - 0 = 0$ then $w_2 \in S$.

Then

$$w_1 + w_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \quad 1 \cdot 1 - 0 \cdot 0 = 1.$$

So $w_1 + w_2 \notin S$.

So S is not a subspace of $M_{2\times 2}(\mathbb{R})$. □

5.2 Linear transformations

VS8HS85. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.

Let $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ be the function given by $T(x) = Ax$ so that

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ 5x_5 + 6x_6 + 7x_7 + 8x_8 \end{pmatrix}.$$

Show that T is a linear transformation.

Solution. Let $u, v \in \mathbb{R}^4$. Then, by the distributive property of matrix multiplication,

$$T(u + v) = A(u + v) = Au + Av = T(u) + T(v).$$

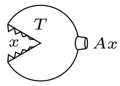
Let $u \in \mathbb{R}^4$ and $c \in \mathbb{R}$. Then, by the associative property of scalar multiplication for matrices,

$$T(cu) = Acu = cAu = cT(u).$$

So T is a linear transformation. $\square$

VS9HS86. Let $t, s \in \mathbb{Z}_{>0}$ and $A \in M_{t\times s}(\mathbb{R})$. Let $T: \mathbb{R}^s \to \mathbb{R}^t$ be the function given by

$$T(x) = Ax.$$



Show that T is a linear transformation.

Solution. Let $u, v \in \mathbb{R}^s$. Then, by the distributive property of matrix multiplication for matrices,

$$T(u+v) = A(u+v) = Au + Av = T(u) + T(v).$$

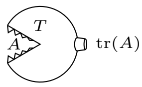
Let $u \in \mathbb{R}^s$ and $c \in \mathbb{R}$. Then, by the associative property of scalar multiplication for matrices,

$$T(cu) = Acu = cAu = cT(u).$$

So T is a linear transformation. $\square$

VS10HS87. Let $n \in \mathbb{Z}_{>0}$ and let $T: M_n(\mathbb{Q}) \to \mathbb{Q}$ by the function given by

$$T(A) = \mathrm{tr}(A).$$



Show that T is a $\mathbb{Q}$-linear transformation.

Solution. Let $A, B \in M_{n\times n}(\mathbb{Q})$. Then

$$\begin{aligned} T(A+B) = \mathrm{tr}(A+B) &= (A+B)_{11} + \cdots + (A+B)_{nn} \\ &= A_{11} + B_{11} + \cdots + A_{nn} + B_{nn} \\ &= A_{11} + \cdots + A_{nn} + +B_{11} + \cdots B_{nn} \\ &= \mathrm{tr}(A) + \mathrm{tr}(B) = T(A) + T(B). \end{aligned}$$

Let $A \in M_{n\times n}(\mathbb{Q})$ and $c \in \mathbb{Q}$. Then

$$\begin{aligned} T(cA) = \mathrm{tr}(cA) &= (cA)_{11} + \cdots + (cA)_{nn} \\ &= cA_{11} + \cdots + cA_{nn} \\ &= c(A_{11} + \cdots + A_{nn}) = c\mathrm{tr}(A) = cT(A). \end{aligned}$$

So T is a linear transformation. $\square$

VS11HS88. Let $T: M_{2\times 2}(\mathbb{Q}) \to \mathbb{Q}$ be the function given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Show that T is a linear transformation.

Solution. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and let $c = 2$. Then

$$T(cA) = \det(2A) = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\cdot 2 - 0\cdot 0 = 4$$

and

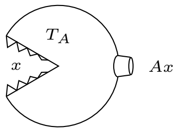
$$cT(A) = 2\det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = 2\cdot(1\cdot 1 - 0\cdot 0) = 2.$$

So this gives an example where $T(cA) \neq cT(A)$.

So T *cannot possibly be a linear transformation.* □

VS12HS130. Let $t, s \in \mathbb{Z}_{>0}$ and $A \in M_{t\times s}(\mathbb{R})$. Let $T_A\colon \mathbb{R}^s \to \mathbb{R}^t$ be the function given by

$$T_A(x) = Ax.$$



Show that T_A is a linear transformation.

Solution. Let $u, v \in \mathbb{R}^s$. Then, by the distributive property of matrix multiplication for matrices,

$$T_A(u+v) = A(u+v) = Au + Av = T_A(u) + T_A(v).$$

Let $u \in \mathbb{R}^s$ and $c \in \mathbb{R}$. Then, by the associative property of scalar multiplication for matrices,

$$T_A(cu) = Acu = cAu = cT_A(u).$$

So T_A is a linear transformation. □

VS13HS131. Let $T\colon V \to W$ be a linear transformation. Assume that T has an inverse function $T^{-1}\colon W \to V$. Show that T^{-1} is a linear transformation.

Solution. Assume $w_1, w_2 \in W$. Then

$$\begin{aligned} T^{-1}(w_1 + w_2) &= T^{-1}(T(T^{-1}(w_1)) + T(T^{-1}(w_2))) \\ &= T^{-1}(T(T^{-1}(w_1) + T^{-1}(w_2)) \\ &= T^{-1}(w_1) + T^{-1}(w_2), \end{aligned}$$

where the first equality is because $T\circ T^{-1} = \mathrm{id}$), the second equality is because T is a linear transformation) and the third equality is because $T^{-1}\circ T = \mathrm{id}$. Assume $w \in W$ and $c \in \mathbb{R}$. Then

$$T^{-1}(cw) = T^{-1}(c\cdot T(T^{-1}(w))) = T^{-1}T(c\cdot T^{-1}(w)) = c\cdot T^{-1}(w).$$

So T^{-1} is a linear transformation. □

VS14HS144. Is the function $T\colon M_2(\mathbb{R}) \to \mathbb{R}$ given by

$$T\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \det\begin{pmatrix}a & b\\ c & d\end{pmatrix} = ad - bc \quad \text{a linear transformation?}$$

Solution. A *linear transformation from* $M_2(\mathbb{R})$ *to* $\mathbb{R}$ is a function $f\colon M_2(\mathbb{R}) \to \mathbb{R}$ such that

(a) If $v_1, v_2 \in M_2(\mathbb{R})$ then $f(v_1 + v_2) = f(v_1) + f(v_2)$,

(b) If $c \in \mathbb{R}$ and $v \in M_2(\mathbb{R})$ then $f(cv) = cf(v)$.

Since

$$1 = T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is not equal to

$$0 = 0 + 0 = T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

then condition (a) does not hold and T is not a linear transformation. □

VS15HS145. Is the function $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3) = (x_2 - 2x_3, 3x_1 + x_3) \qquad \text{a linear transformation?}$$

Solution. A *linear transformation from* $\mathbb{R}^3$ *to* $\mathbb{R}^2$ is a function $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ such that

(a) If $u, v \in \mathbb{R}^3$ then $f(u+v) = f(u) + f(v)$,

(b) If $c \in \mathbb{R}$ and $v \in \mathbb{R}^3$ then $f(cv) = cf(v)$.

(a) Assume $u, v \in \mathbb{R}^3$ with $u = |u_1, u_2, u_3\rangle$ and $v = |v_1, v_2, v_3\rangle$. Then

$$\begin{aligned} T(|u_1, u_2, u_3\rangle + |v_1, v_2, v_3\rangle &= T(|u_1 + v_1, u_2 + v_2, u_3 + v_3\rangle) \\ &= |(u_2 + v_2 - 2(u_3 + v_3), 3(u_1 + v_1) + (u_3 + v_3)\rangle \\ &= |u_2 - 2u_3 + v_2 - 2v_3, 3u_1 + u_3 + 3v_1 + v_3\rangle \\ &= |u_2 - 2u_3, 3u_1 + u_3\rangle + |v_2 - 2v_3, 3v_1 + v_3\rangle \\ &= T(|u_1, u_2, u_3\rangle) + T(|v_1, v_2, v_3\rangle) \end{aligned}$$

(b) Assume $c \in \mathbb{R}$ and $u \in \mathbb{R}^3$ with $u = |u_1, u_2, u_3\rangle$. Then

$$\begin{aligned} T(c \cdot |u_1, u_2, u_3\rangle) = T(|cu_1, cu_2, cu_3\rangle) &= |cu_2 - 2cu_3, 3cu_1 + cu_3\rangle \\ &= c|u_2 - 2u_3, 3u_1 + u_3\rangle = cT(|u_1, u_2, u_3\rangle). \end{aligned}$$

So T is a linear transformation. □

5.3 Linear independence, span and bases

VS16HS89. Let V be a $\mathbb{Q}$-vector space and let $v_1, v_2, v_3, v_4, v_5 \in V$. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$. Show that $\mathbb{Q}$-span(S) is a subspace of V.

Solution. (a) Since $0 = 0v_1 + 0v_2 + 0v_3 + 0v_4 + 0v_5$ then $0 \in \mathbb{Q}$-span(S).

(b) Assume $a = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 \in \mathbb{Q}$-span$(S)$ and $b = b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4 + b_5v_5 \in \mathbb{Q}$-span$(S)$. Then

$$\begin{aligned} a + b &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 \\ &\quad + b_1v_1 + b_2v_2 + b_3v_3 + b_4v_4 + b_5v_5 \\ &= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + (a_3 + b_3)v_3 \\ &\quad + (a_4 + b_4)v_4 + (a_5 + b_5)v_5. \end{aligned}$$

So $a + b \in \mathbb{Q}$-span(S).

(c) Assume $a = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 \in \mathbb{Q}\text{-span}(S)$ and $c \in \mathbb{Q}$. Then

$$\begin{aligned} ca &= c(a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5) \\ &= ca_1v_1 + ca_2v_2 + ca_3v_3 + ca_4v_4 + ca_5v_5 \in \mathbb{Q}\text{-span}(S). \end{aligned}$$

So $\mathbb{Q}$-span(S) is a subspace of V. □

VS17HS90. In $\mathbb{R}[x]_{\le 2}$, is $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$?

Solution. By definition $\mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$
$= \{c_1(1 + x + x^2) + c_2(3 + x^2) \mid c_1c_2 \in \mathbb{R}\}$.

So we need to show that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1(1 + x + x^2) + c_2(3 + x^2) = 1 - 2x - x^2.$$

So we need to show that the system $\begin{array}{l} c_1 + 3c_2 = 1, \\ c_1 + 0c_2 = -2, \\ c_1 + c_2 = -1, \end{array}$ has a solution.

The second equation gives $c_1 = -2$ and then $c_2 = -1 - c_1 = 1 + 2 = 3$. Since the equation $c_1 + 3c_2 = 1$ also works when $c_1 = -2$ and $c_2 = 3$ then $c_1 = -2$, $c_2 = 1$ is a solution to this system.

Alternatively, the solution can be found by row reduction as follows. In matrix form the equations are

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

So $c_1 = -2$ and $c_2 = 1$ is a solution.
So $-2(1+x+x^2) + (3+x^2) = 1-2x-x^2$.
So $1-2x-x^2 \in \mathbb{R}\text{-span}\{1+x+x^2, 3+x^2\}$.
So $1-2x-x^2$ is a linear combination of $1+x+x^2$ and $3+x^2$ and

$$1-2x-x^2 \in \mathbb{R}\text{-span}\{1+x+x^2, 3+x^2\}.$$

□

VS18HS91. Let S be the subset of $\mathbb{R}[x]_{\le 2}$ given by

$$S = \{1+2x, 1+5x+3x^2, x+x^2\}. \qquad \text{Show that } \mathbb{R}\text{-span}(S) = \mathbb{R}[x]_{\le 2}.$$

Proof. By definition

$$\mathbb{R}\text{-span}(S) = \{c_1(1+2x) + c_2(1+5x+3x^2) + c_3(x+x^2) \mid c_1, c_2, c_3 \in \mathbb{R}\}.$$

To show: (a) $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^3$
(b) $\mathbb{R}^3 \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}^3$ and $\mathbb{R}^3$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}[x]_{\le 2}$.
(b) To show: $\mathbb{R}[x]_{\le 2} \subseteq \mathbb{R}\text{-span}(S)$.
To show: $\mathbb{R}\text{-span}\{1, x, x^2\} \subseteq \text{span}(S)$.
Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication,
To show: $\{1, x, x^2\} \subseteq \mathbb{R}\text{-span}(S)$.

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{aligned} c_1(1+2x) + c_2(1+5x+3x^2) + c_3(x+x^2) &= 1+0x+x^2, \\ d_1(1+2x) + d_2(1+5x+3x^2) + d_3(x+x^2) &= 0+x+0x^2, \\ r_1(1+2x) + r_2(1+5x+3x^2) + r_3(x+x^2) &= 0+0x+x^2. \end{aligned}$$

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Since the bottom row on the left hand side is all 0 and the bottom row on the right hand sides is not all 0 then there *do not exist* $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{1, x, x^2\} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\text{span}(S) \neq \mathbb{R}[x]_{\leq 2}$. □

VS19HS146. In $\mathbb{R}^3$, is $|1,2,3\rangle \in \mathbb{R}\text{-span}\{|1,-1,2\rangle, |-1,1,2\rangle\}$?

Solution. By definition,

$$\mathbb{R}\text{-span}\{|1,-1,2\rangle, |-1,1,2\rangle\} = \{c_1|1,-1,2\rangle + c_2|-1,1,2\rangle \mid c_1, c_2 \in \mathbb{R}\}.$$

So we need to show that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$|1,2,3\rangle = c_1|1,-1,2\rangle + c_2|-1,1,2\rangle.$$

So we need to show that the system $\begin{matrix} c_1 - c_2 = 1, \\ -c_1 + c_2 = 2, \\ 2c_1 + 2c_2 = 3, \end{matrix}$ has a solution.

In matrix form the equations are $\begin{pmatrix} 2 & 2 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 2 & 2 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

Already this gives an equation $0c_1 + 0c_2 = 3$, which has no solution.

So $|1,2,3\rangle \notin \mathbb{R}\text{-span}\{|1,-1,2\rangle$ and $|-1,1,2\rangle\}$.

So $|1,2,3\rangle$ is not a linear combination of $|1,-1,2\rangle$ and $|-1,1,2\rangle$.

So $|1,2,3\rangle \notin \mathbb{R}\text{-span}\{|1,-1,2\rangle, |-1,1,2\rangle\}$. □

VS20HS147. In $\mathbb{R}[x]_{\leq 2}$, is $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$?

Solution. By definition,

$$\mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\} = \{c_1(1 + x + x^2) + c_2(3 + x^2) \mid c_1 c_2 \in \mathbb{R}\}.$$

So we need to show that there exist $c_1, c_2 \in \mathbb{R}$ such that

$$c_1(1 + x + x^2) + c_2(3 + x^2) = 1 - 2x - x^2.$$

So we need to show that the system $\begin{matrix} c_1 + 3c_2 = 1, \\ c_1 + 0c_2 = -2, \\ c_1 + c_2 = -1, \end{matrix}$ has a solution.

In matrix form the equations are

$$\begin{pmatrix} 1 & 3 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & -1\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 3\\ 1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}1\\ -1\\ -1\end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}0 & 1 & 0\\ 1 & -1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 1\\ 0 & 2\\ 0 & -1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}-1\\ 2\\ -1\end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & -2\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 1\\ 0 & -1\\ 0 & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}-1\\ -1\\ 0\end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 1\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}-1\\ 1\\ 0\end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}-2\\ 1\\ 0\end{pmatrix}.$$

So $c_1 = -2$ and $c_2 = 1$ is a solution.
So $-2(1+x+x^2) + (3+x^2) = 1 - 2x - x^2$.
So $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1+x+x^2, 3+x^2\}$.
So $1 - 2x - x^2$ is a linear combination of $1+x+x^2$ and $3+x^2$. □

VS21HS148. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(1,1,1),(2,2,2),(3,3,3)\}. \qquad \text{Determine } \mathbb{R}\text{-span}(S).$$

Solution. In this case

$$\begin{aligned}\mathbb{R}\text{-span}(S) &= \{c_1\,|1,1,1\rangle + c_2\,|2,2,2\rangle + c_3\,|3,3,3\rangle \mid c_1,c_2,c_3\in\mathbb{R}\}\\ &= \{c_1\,|1,1,1\rangle + 2c_2\,|1,1,1\rangle + 3c_3\,|1,1,1\rangle \mid c_1,c_2,c_3\in\mathbb{R}\}\\ &= \{(c_1+2c_2+3c_3)\,|1,1,1\rangle \mid c_1,c_2,c_3\in\mathbb{R}\}\\ &= \{t\,|1,1,1\rangle \mid t\in\mathbb{R}\}\\ &= \{|t,t,t\rangle \mid t\in\mathbb{R}\}.\end{aligned}$$

□

VS22HS149. Let S be the subset of $\mathbb{R}^2$ given by

$$S = \{|1,-1\rangle, |2,4\rangle\}. \qquad \text{Show that } \mathbb{R}\text{-span}(S) = \mathbb{R}^2.$$

Solution. *Proof.* By definition, $\mathbb{R}\text{-span}(S) = \{c_1|1,-1\rangle + c_2|2,4\rangle \mid c_1, c_2 \in \mathbb{R}\}$.
To show: (a) $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$
(b) $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.
(a) Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar mutliplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$.
(b) To show: $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.
To show: $\mathbb{R}\text{-span}\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.
Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication, we can show $\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.
To show: There exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that

$$c_1|1,-1\rangle + c_2|2,4\rangle = |1,0\rangle \quad \text{and} \quad d_1|1,-1\rangle + d_2|2,4\rangle = |0,1\rangle.$$

To show: There exist $c_1, c_2, d_1, d_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} c_1 & d_1 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$\tfrac{2}{3}|1,-1\rangle + \tfrac{1}{6}|2,4\rangle = |1,0\rangle, \text{ and}$$

$$-\tfrac{1}{3}|1,-1\rangle + \tfrac{1}{6}|2,4\rangle = |0,1\rangle.$$

So $|1,0\rangle \in \mathbb{R}\text{-span}(S)$ and $|0,1\rangle \in \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}\text{-span}\{|1,0\rangle, |0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}\text{-span}(S) = \mathbb{R}^2$. □

VS23HS150. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{|1,2,0\rangle, |1,5,3\rangle, |0,1,1\rangle\}. \qquad \text{Show that } \text{span}(S) = \mathbb{R}^3.$$

Solution. *Proof.* By definition,

$$\mathbb{R}\text{-span}(S) = \{c_1|1,2,0\rangle + c_2|1,5,3\rangle + c_3|0,1,1\rangle \mid c_1, c_2, c_3 \in \mathbb{R}\}.$$

To show: (a) $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^3$
(b) $\mathbb{R}^3 \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}^3$ and $\mathbb{R}^3$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^3$.
(b) To show: $\mathbb{R}^3 \subseteq \mathbb{R}\text{-span}(S)$.
To show: $\mathbb{R}\text{-span}\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \subseteq \text{span}(S)$.
Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication,
To show: $\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \subseteq \mathbb{R}\text{-span}(S)$.
To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{aligned} c_1|1,2,0\rangle + c_2|1,5,3\rangle + c_3|0,1,1\rangle &= |1,0,0\rangle, \\ d_1|1,2,0\rangle + d_2|1,5,3\rangle + d_3|0,1,1\rangle &= |0,1,0\rangle, \\ r_1|1,2,0\rangle + r_2|1,5,3\rangle + r_3|0,1,1\rangle &= |0,0,1\rangle, \end{aligned}$$

To show: There exist $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Since the bottom row on the left hand side is all 0 and the bottom row on the right hand sides is not all 0 then there *do not exist* $c_1, c_2, c_3, d_1, d_2, d_3, r_1, r_2, r_3 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 5 & 1 \\ 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \\ c_3 & d_3 & r_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\} \not\subseteq \mathbb{R}\text{-span}(S)$.

So $\mathrm{span}(S) \neq \mathbb{R}^2$. □

VS24HS151. Let S be the subset of $\mathbb{R}[x]_{\le 2}$ given by

$$S = \{1 + x + x^2, x^2\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}[x]_{\le 2}.$$

Solution. *Proof.* By definition,

$$\mathbb{R}\text{-span}(S) = \{c_1(1 + x + x^2) + c_2x^2 \mid c_1, c_2 \in \mathbb{R}\}.$$

To show: (a) $\mathrm{span}(S) \subseteq \mathbb{R}[x]_{\le 2}$
(b) $\mathbb{R}[x]_{\le 2} \subseteq \mathbb{R}\text{-span}(S)$.

(a) Since $S \subseteq \mathbb{R}[x]_{\le 2}$ and $\mathbb{R}[x]_{\le 2}$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}[x]_{\le 2}$.
(b) To show: $\mathbb{R}[x]_{\le 2} \subseteq \mathbb{R}\text{-span}(S)$.
To show: $\mathbb{R}\text{-span}\{1, x, x^2\} \subseteq \mathbb{R}\text{-span}(S)$.
Since $\mathbb{R}\text{-span}(S)$ is closed under addition and scalar multiplication,
To show: $\{1, x, x^2\} \subseteq \mathbb{R}\text{-span}(S)$.
To show: There exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$c_1(1 + x + x^2) + c_2x^2 = 1, \quad d_1(1 + x + x^2) + d_2x^2 = x,$$

and

$$r_1(1 + x + x^2) + r_2x^2 = x^2.$$

To show: There exist $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ to get } \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the top row on the left hand side is all 0 and the top row on the right hand sides is not all 0 then there *do not exist* $c_1, c_2, d_1, d_2, r_1, r_2 \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 & d_1 & r_1 \\ c_2 & d_2 & r_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

So $\{1, x, x^2\} \not\subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}\text{-span}\{1, x, x^2\} \not\subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}[x]_{\le 2} \not\subseteq \mathbb{R}\text{-span}(S)$.
So $\mathbb{R}\text{-span}(S) \ne \mathbb{R}[x]_{\le 2}$. □

VS25HS152. Let S be the subset of $\mathbb{C}^3$ given by

$$S = \{|2i, -1, 1\rangle, |-6, -3i, 3i\rangle\}. \qquad \text{Is } S\ \mathbb{C}\text{-linearly independent?}$$

Solution. To show: If $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|-6, -3i, 3i\rangle = |0, 0, 0\rangle$ then $c_1 = 0$, $c_2 = 0$.
Assume $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|-6, -3i, 3i\rangle = |0, 0, 0\rangle$.
Then

$$\begin{aligned} 2ic_1 - 6c_2 &= 0, \\ -c_1 - 3ic_2 &= 0, \\ c_1 + 3ic_2 &= 0, \end{aligned} \quad \text{or equivalently} \quad \begin{pmatrix} 2i & -6 \\ -1 & -3i \\ 1 & 3i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has solutions

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3i \\ 1 \end{pmatrix}, \qquad \text{with } t \in \mathbb{C}.$$

So $c_1 = 0$, $c_2 = 0$ is not the only solution.
So S is not linearly independent. □

VS26HS153. Let B be the subset of $\mathbb{C}^3$ given by

$$B = \{|2i, -1, 1\rangle, |4, 0, 2\rangle\}. \qquad \text{Is } B \text{ linearly independent?}$$

Solution. To show: If $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|4, 0, 2\rangle = |0, 0, 0\rangle$ then $c_1 = 0$, $c_2 = 0$.
Assume $c_1, c_2 \in \mathbb{C}$ and $c_1\,|2i, -1, 1\rangle + c_2|4, 0, 2\rangle = |0, 0, 0\rangle$
Then

$$\begin{aligned} 2ic_1 + 4c_2 &= 0, \\ -c_1 + 0c_2 &= 0, \\ c_1 + 2c_2 &= 0, \end{aligned} \quad \text{or equivalently} \quad \begin{pmatrix} 2i & 4 \\ -1 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has only one solution $c_1 = 0$, $c_2 = 0$.
So S is linearly independent. $\square$

VS27HS154. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(2,0,0), (6,1,7), (2,-1,2)\}. \qquad \text{Is } S \text{ linearly independent?}$$

Solution. To show:
If $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1\,|2,0,0\rangle + c_2|6,1,7\rangle + c_3|2,-1,2\rangle = |0,0,0\rangle$
then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
Assume $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1\,|2,0,0\rangle + c_2|6,1,7\rangle + c_3|2,-1,2\rangle = |0,0,0\rangle$.
Then

$$\begin{aligned} 2c_1 + 6c_2 + 2c_3 &= 0, \\ c_2 - c_3 &= 0, \\ 7c_2 + 2c_3 &= 0, \end{aligned} \quad \text{or equivalently} \quad \begin{pmatrix} 2 & 6 & 2 \\ 0 & 1 & -1 \\ 0 & 7 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has only one solution: $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
So S is linearly independent. $\square$

VS28HS155. Let S be the subset of $\mathbb{R}[x]_{\le 2}$ given by

$$S = \{1 + 2x + 5x^2, 1 + x + x^2, 1 + 2x + 3x^2\}. \quad \text{Is } S \text{ a basis of } \mathbb{R}[x]_{\le 2}?$$

Solution. To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1(1 + 2x + 5x^2) + c_2(1 + x + x^2) + c_3(1 + 2x + 3x^2) = 0$
then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
Assume $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1(1 + 2x + 5x^2) + c_2(1 + x + x^2) + c_3(1 + 2x + 3x^2) = 0$.
Then

$$\begin{aligned} c_1 + c_2 + c_3 &= 0, \\ 2c_1 + c_2 + 2c_3 &= 0, \\ 5c_1 + c_2 + 3c_2 &= 0, \end{aligned} \quad \text{or, equivalently,} \quad \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 5 & 1 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has only one solution: $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
So S is linearly independent.

Since $\dim(\mathbb{R}[x]_{\le 2}) = 3$ and S contains 3 linearly independent elements then B is a basis for $\mathbb{R}[x]_{\le 2}$. $\square$

VS29HS156. Let S be the subset of $M_2(\mathbb{R})$ given by

$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\}. \qquad \text{Is } S \text{ linearly independent?}$$

Solution. To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
Assume $c_1, c_2, c_3 \in \mathbb{R}$ and $c_1 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Then

$$\begin{matrix} c_1 - 2c_2 + c_3 = 0, \\ 3c_1 + c_2 + 10c_3 = 0, \\ c_1 + c_2 + 4c_3 = 0, \\ c_1 - c_2 + 2c_3 = 0, \end{matrix} \quad \text{or, equivalently,} \quad \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Skipping the row reduction steps (DON'T skip the row reduction steps on an exam or an assignment!), this system has solutions

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ -1 \end{pmatrix}, \qquad \text{with } t \in \mathbb{R}.$$

So $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ is not the only solution.
So S is not linearly independent.

Here is a check that $c_1 = -3$, $c_2 = 1$, $c_3 = -1$ is a solution:

$$-3\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = -3\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -3 & -9 \\ -3 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

□

VS30HS157. Is $S = \{(1, -1), (2, 4)\}$ a basis of $\mathbb{R}^2$?

Solution. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}. \qquad \text{Then} \qquad A^{-1} = \tfrac{1}{6}\begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{gives} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So S is linearly independent.

If $|a, b\rangle \in \mathbb{R}^2$ then $|a, b\rangle = c_1|1, -1\rangle + c_2|2, 4\rangle$, where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{2}{3}a - \frac{1}{3}b \\ \frac{1}{6}a + \frac{1}{6}b \end{pmatrix}.$$

So $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$. Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$. So $\mathbb{R}\text{-span}(S) = \mathbb{R}^2$.

So S is a basis of $\mathbb{R}^2$. □

VS31HS158. Is $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ a basis of $\{A \in M_2(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$?

Solution. If $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_2\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{matrix} c_1 = 0, \\ c_2 = 0, \\ c_3 = 0. \end{matrix}$$

So S is linearly independent.

Then

$$\begin{aligned}
&\{A \in M_2(\mathbb{R}) \mid \mathrm{tr}(A) = 0\} \\
&= \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}, a_{11} + a_{22} = 0 \right\} \\
&= \left\{ \begin{pmatrix} c_1 & c_2 \\ c_3 & -c_1 \end{pmatrix} \mid c_1, c_2, c_3 \in \mathbb{R} \right\} \\
&= \left\{ c_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + c_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;\Big|\; c_1, c_2, c_3 \in \mathbb{R} \right\} \\
&= \mathrm{span}(S).
\end{aligned}$$

So S is a basis of $\{A \in M_2(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$. □

VS32HS159. Is

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \qquad \text{a basis of } M_2(\mathbb{R})?$$

Solution. Since $E_{11}, E_{12}, E_{21}, E_{22}$ is a basis of $M_2(\mathbb{R})$ then $\dim(M_2(\mathbb{R})) = 4$.
Since S contains only 3 elements then S is not a basis of $M_2(\mathbb{R})$. □

VS33HS92. Is $S = \{(1, -1), (2, 4)\}$ a basis of $\mathbb{R}^2$?

Solution. Let

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}. \qquad \text{Then} \qquad A^{-1} = \tfrac{1}{6} \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

So

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{gives} \qquad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

So S is linearly independent.

If $|a, b\rangle \in \mathbb{R}^2$ then $|a, b\rangle = c_1|1, -1\rangle + c_2|2, 4\rangle$, where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{2}{3}a - \frac{1}{3}b \\ \frac{1}{6}a + \frac{1}{6}b \end{pmatrix}.$$

So $\mathbb{R}^2 \subseteq \mathbb{R}\text{-span}(S)$. Since $S \subseteq \mathbb{R}^2$ and $\mathbb{R}^2$ is closed under addition and scalar multiplication then $\mathbb{R}\text{-span}(S) \subseteq \mathbb{R}^2$. So $\mathbb{R}\text{-span}(S) = \mathbb{R}^2$.

So S is a basis of $\mathbb{R}^2$. □

VS34HS93. Let S be the subset of $M_2(\mathbb{R})$ given by

$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\}. \qquad \text{Is } S \text{ linearly independent?}$$

Solution. To show: If $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1 \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3 \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

then $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Suppose an oracle tells you to try (or you guess) $c_1 = -3$, $c_2 = 1$, $c_3 = -1$ and then you verify that

$$-3\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = -3\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -3 & -9 \\ -3 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that you don't have to have c_1, c_2 and c_3 all 0 to get a zero linear combination. So S is not linearly independent.

If you have no oracle, or are not a good guesser, then proceed as follows.

Assume $c_1, c_2, c_3 \in \mathbb{R}$ and

$$c_1\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} + c_2\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} + c_3\begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} c_1 - 2c_2 + c_3 &= 0, \\ 3c_1 + c_2 + 10c_3 &= 0, \\ c_1 + c_2 + 4c_3 &= 0, \\ c_1 - c_2 + 2c_3 &= 0, \end{aligned} \quad \text{or, equivalently,} \quad \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 10 \\ 1 & 1 & 4 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 3 & 1 & 10 \\ 1 & -1 & 2 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 2 \\ 0 & 4 & 4 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 4 & 4 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

This gives the system

$$\begin{aligned} c_1 + 3c_3 &= 0, \\ c_2 + c_3 &= 0, \end{aligned} \quad \text{which is} \quad \begin{aligned} c_1 &= -3c_3 \\ c_2 &= -c_3, \\ c_3 &= c_3, \end{aligned}$$

which has solutions

$$\left\{ c_3 \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; c_3 \in \mathbb{R} \right\} = \mathbb{R}\text{-span}\left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

So $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ is not the only solution.
This means that you don't have to have c_1, c_2 and c_3 all 0 to get a zero linear combination.
So S is not linearly independent. □

VS35HS83. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ be given by

$$v_1 = |1, 2, 3\rangle, \quad v_2 = |3, 6, 9\rangle, \quad v_3 = |-1, 0, -2\rangle, \quad v_4 = |1, 4, 4\rangle.$$

(a) Is $\{v_1, v_2, v_3, v_4\}$ linearly independent?
(b) Express v_2 and v_4 as linear combinations of v_1 and v_3.
(c) Is $\{v_1, v_3\}$ linearly independent?

Solution. (a) Since $v_2 = 3v_1$ then $0 = 3v_1 - v_2 = 3v_1 - v_2 + 0v_3 + -v_4$.
So $c_1 = 3$, $c_2 = -1$, $c_3 = 0$, $c_4 = 0$ is a case that gives $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$.
So $\{v_1, v_2, v_3, v_4\}$ is not linearly independent.

(b) Since $v_2 = 3v_1 + 0v_3$ then $v_2 \in \mathbb{R}\text{-span}\{v_1, v_3\}$.
Since $v_1 + v_3 = (0, 2, 1)$ and $v_1 + |0, 2, 1\rangle = |1, 4, 4\rangle$.
So $v_4 = 2v_1 + v_3$. So $v_4 \in \mathbb{R}\text{-span}\{v_1, v_3\}$.

(c) To show: If $c_1, c_2 \in \mathbb{R}$ and $c_1|1, 2, 3\rangle + c_2|-1, 0, 2\rangle = |0, 0, 0\rangle$
then $c_1 = 0$ and $c_2 = 0$.

Assume $c_1, c_2 \in \mathbb{R}$ and $c_1|1, 2, 3\rangle + c_2|-1, 0, 2\rangle = |0, 0, 0\rangle$.
Then

$$\begin{aligned} c_1 - c_2 &= 0, \\ 2c_1 + 0c_2 &= 0, \\ 3c_1 + 2c_2 &= 0. \end{aligned}$$

The first equation gives $c_1 = c_2$ and the second equation gives $2c_1 = 0$ so that $c_2 = c_1 = 0$. This system has

$$\text{only one solution:} \qquad c_1 = 0, \quad c_2 = 0.$$

So $\{v_1, v_3\}$ is linearly independent. $\square$

VS36AQ1. Let $\mathbb{R}[x]_{\le 2} = \{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$. Let $t \in \mathbb{R}$ and let

$$g_1(x) = 1, \qquad g_2(x) = x + t, \qquad g_3(x) = (x+t)^2.$$

Prove that $\{g_1, g_2, g_3\}$ is a basis of $\mathbb{R}[x]_{\le 2}$.

Solution. To show: $\{g_1, g_2, g_3\}$ is a basis of $\mathbb{R}[x]_{\le 2}$.
To show: (a) $\{g_1, g_2, g_3\}$ is linearly independent.
(b) $\mathrm{span}\{g_1, g_2, g_3\} = P_2$.

(a) Assume $c_1, c_2, c_3 \in \mathbb{R}$ such that $c_1g_1 + c_2g_2 + c_3g_3 = 0$.
To show: $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$.
We know

$$0 = c_1 \cdot 1 + c_2(x+t) + c_3(x^2 + 2tx + t^2) = (c_1 + c_2t + c_3t^2) + (c_2 + 2tc_3)x + c_3x^2.$$

Comparing coefficients on each side gives

$$\begin{aligned} c_3 &= 0, \\ c_2 + 2tc_3 &= 0, \\ c_1 + tc_2 + t^2c_3 &= 0, \end{aligned} \qquad \text{so that} \qquad \begin{aligned} c_3 &= 0, \\ c_2 + 0 &= 0, \\ c_1 + 0 + 0 &= 0. \end{aligned}$$

So $\{g_1, g_2, g_3\}$ is linearly independent.

(b) To show: (ba) $\mathrm{span}\{g_1, g_2, g_3\} \subseteq P_2$,
(bb) $P_2 \subseteq \mathrm{span}\{g_1, g_2, g_3\}$.

(ba) Since $g_1, g_2, g_3 \in P_2$ and P_2 is closed under addition and scalar multiplication then $\mathrm{span}\{g_1, g_2, g_3\} \subseteq P_2$.

(bb) To show: If $a_0, a_1, a_2 \in \mathbb{R}$ then $a_0 + a_1x + a_2x^2 \in \mathrm{span}\{g_1, g_2, g_3\}$.
Assume $a_0, a_1, a_2 \in \mathbb{R}$.
Then

$$\begin{aligned} a_0 + a_1x + a_2x^2 &= a_0 + a_1(x + 1 - t) + a_2((x+t) - t)^2 \\ &= a_0 \ + a_1((x+t) - t) + a_2(x+t)^2 - 2t(x+t) + t^2) \\ &= a_0 - ta_1 + t^2a_2 + (a_1 - 2ta_2)(x+t) + a_2(x+t)^2 \\ &= (a_0 - ta_1 + t^2a_2)g_1 + (a_1 - 2ta_2)g_2 + a_2g_3 \in \mathrm{span}\{g_1, g_2, g_3\}. \end{aligned}$$

Since $\mathrm{span}\{g_1, g_2, g_3\} \subseteq P_2$ and $P_2 \subseteq \mathrm{span}\{g_1, g_2, g_3\}$
then $\mathrm{span}\{g_1, g_2, g_3\} = P_2$.

Since $\{g_1, g_2, g_3\}$ is linearly independent and $\mathrm{span}\{g_1, g_2, g_3\} = P_2$
then $\{g_1, g_2, g_3\}$ is a basis of P_2. $\square$