1 Matrices and operations

1.1 Addition, scalar multiplication and multiplication

MO1AS1. Calculate

$$\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \end{pmatrix}.$$

Solution. Addition of matrices is componentwise, entry by entry.

$$\begin{aligned} \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \end{pmatrix} &= \begin{pmatrix} 3+1 & 2+6 & 2+3 & 1+4 \\ 0+5 & 7+6 & 7-9 & 5+8 \\ 1-5 & 3.2-2 & 5+i\sqrt{2} & -1+0 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 8 & 5 & 5 \\ 5 & 13 & -2 & 13 \\ -4 & 1.2 & 5+i\sqrt{2} & -1 \end{pmatrix} \end{aligned}$$

□

MO2AS2. Calculate

$$\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \\ -1 & -2 & -3 & -4 \end{pmatrix}.$$

Solution. Addition of matrices of different sizes is not defined.

$$\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \\ -1 & -2 & -3 & -4 \end{pmatrix} \qquad \text{is not defined.}$$

□

MO3AS3. Explain the meaning of 0 in the following expression and compute the sum.

$$0 + \begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -2 & i\sqrt{2} \\ -4.1 & -3 \end{pmatrix}$$

Solution.

$$0 + \begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -2 & i\sqrt{2} \\ -4.1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -2 & i\sqrt{2} \\ -4.1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -2 & i\sqrt{2} \\ -4.1 & -3 \end{pmatrix}$$

□

MO4AS4. Explain the meaning of the negative before the first matrix in the following expression and compute the sum.

$$-\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix}$$

Solution.

$$\begin{aligned} -\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} &= \begin{pmatrix} -3 & -6 & -2 & -1 \\ 0 & -7 & 9 & -5 \\ -1 & -3.2 & -5 & 1 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

□

MO5AS5. Calculate

$$2i \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & 3.2 & i\sqrt{2} & 0 \end{pmatrix}.$$

Solution. Scalar multiplication by c multiplies each entry by c.

$$\begin{aligned} 2i \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & 3.2 & i\sqrt{2} & 0 \end{pmatrix} &= \begin{pmatrix} 2i \cdot 1 & 2i \cdot 2 & 2i \cdot 3 & 2i \cdot 4 \\ 2i \cdot 5 & 2i \cdot 6 & 2i \cdot 7 & 2i \cdot 2 \\ 2i \cdot (-5) & 2i \cdot 3.2 & 2i \cdot i\sqrt{2} & 2i \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 2i & 4i & 6i & 8i \\ 10i & 12i & 14i & 4i \\ -10i & 6.4i & -2\sqrt{2} & 0 \end{pmatrix} \end{aligned}$$

□

MO6AS6. Calculate

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2i \end{pmatrix}.$$

Solution. A 1×4 matrix multiplied by a 4×1 matrix is a 1×1 matrix.

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2i \end{pmatrix} = (4 \cdot (-1) + 2 \cdot 2 + 6 \cdot 0 + 3 \cdot (-2i)) = (-4 + 4 + 0 - 6i) = (-6i).$$

□

MO7AS7. Calculate

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & 0 & -2i \end{pmatrix}.$$

Solution. Multiplication of matrices is not defined unless the number of columns of the first matrix is the same as the number of rows of the second matrix.

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & 0 & -2i \end{pmatrix} \quad \text{is not defined.}$$

□

MO8AS8. Using that $i^2 = -1$, compute

$$\begin{pmatrix} 4 & 2 & 6 & 3 \\ 1 & 2 & -3 & 2i \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ -2i & 0 & 1 \end{pmatrix}.$$

Solution. Using that $i^2 = -1$,

$$\begin{aligned} &\begin{pmatrix} 4 & 2 & 6 & 3 \\ 1 & 2 & -3 & 2i \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ -2i & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4\cdot(-1)+2\cdot 2+6\cdot 0+3\cdot(-2i) & 4\cdot 0+2\cdot 0+6\cdot 0+3\cdot 0 & 4\cdot 1+2\cdot 1+6\cdot 1+3\cdot 1 \\ 1\cdot(-1)+2\cdot 2+(-3)\cdot 0+2i\cdot(-2i) & 1\cdot 0+2\cdot 0+(-3)\cdot 0+2i\cdot 0 & 1\cdot 1+2\cdot 1+(-3)\cdot 1+2i\cdot 1 \end{pmatrix} \\ &= \begin{pmatrix} -6i & 0 & 15 \\ 7 & 0 & 2i \end{pmatrix}. \end{aligned}$$

□

MO9AS9. Using $i^2 = -1$, compute

$$\begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 6 & -3 \\ 3 & 2i \end{pmatrix}.$$

Solution. Using $i^2 = -1$,

$$\begin{aligned} \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 6 & -3 \\ 3 & 2i \end{pmatrix} &= \begin{pmatrix} -1\cdot 4+2\cdot 2+0\cdot 6-2i\cdot 3 & -1\cdot 1+2\cdot 2+0\cdot(-3)-2i\cdot 2i \\ 0\cdot 4+0\cdot 2+0\cdot 6+0\cdot 3 & 0\cdot 1+0\cdot 2+0\cdot(-3)+0\cdot 2i \\ 1\cdot 4+1\cdot 2+1\cdot 6+1\cdot 3 & 1\cdot 1+1\cdot 2+1\cdot(-3)+1\cdot 2i \end{pmatrix} \\ &= \begin{pmatrix} -6i & 7 \\ 0 & 0 \\ 15 & 2i \end{pmatrix} \end{aligned}$$

□

MO10AS10. Explain the meaning of 1 and compute the products

$$1 \cdot \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot 1.$$

Solution.

$$1\cdot\begin{pmatrix}-1&2&0&-2i\\0&0&0&0\\1&1&1&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&2&0&-2i\\0&0&0&0\\1&1&1&1\end{pmatrix}=\begin{pmatrix}-1&2&0&-2i\\0&0&0&0\\1&1&1&1\end{pmatrix}$$

and

$$\begin{pmatrix}-1&2&0&-2i\\0&0&0&0\\1&1&1&1\end{pmatrix}\cdot 1=\begin{pmatrix}-1&2&0&-2i\\0&0&0&0\\1&1&1&1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}-1&2&0&-2i\\0&0&0&0\\1&1&1&1\end{pmatrix}$$

MO11AS11. Compute the following products.

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&c&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&-c&0\\0&0&0&0&1\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&-c&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&c&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}.$$

Solution. The inverse of the 5×5 matrix $s_3(c)$ is verified by

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&c&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&-c&0\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}=1$$

and

$$\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&0&1&0\\0&0&1&-c&0\\0&0&0&0&1\end{pmatrix}\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&c&1&0\\0&0&1&0&0\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}=1.$$

□

MO12AS12. Compute the following products

$$\begin{pmatrix}4&0&0&0&0\\0&\frac12&0&0&0\\0&0&\frac78&0&0\\0&0&0&1&0\\0&0&0&0&5\end{pmatrix}\begin{pmatrix}\frac14&0&0&0&0\\0&2&0&0&0\\0&0&\frac87&0&0\\0&0&0&1&0\\0&0&0&0&\frac15\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}\frac14&0&0&0&0\\0&2&0&0&0\\0&0&\frac87&0&0\\0&0&0&1&0\\0&0&0&0&\frac15\end{pmatrix}\begin{pmatrix}4&0&0&0&0\\0&\frac12&0&0&0\\0&0&\frac78&0&0\\0&0&0&1&0\\0&0&0&0&5\end{pmatrix}.$$

Solution. The inverse of a diagonal matrix is verified by

$$\begin{pmatrix}4&0&0&0&0\\0&\frac12&0&0&0\\0&0&\frac78&0&0\\0&0&0&1&0\\0&0&0&0&5\end{pmatrix}\begin{pmatrix}\frac14&0&0&0&0\\0&2&0&0&0\\0&0&\frac87&0&0\\0&0&0&1&0\\0&0&0&0&\frac15\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}=1$$

and

$$\begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{7} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{8} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1.$$

This verifies that $d(1, \frac{1}{2}, \frac{7}{8}, 1, 5)d(\frac{1}{4}, 2, \frac{8}{7}, 1, \frac{1}{5}) = 1$ and $d(\frac{1}{4}, 2, \frac{8}{7}, 1, \frac{1}{5})d(1, \frac{1}{2}, \frac{7}{8}, 1, 5) = 1$ □

MO13AS13. Compute the following products.

$$\begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution. The inverse of the 5×5 matrix $x_{14}(7)$ is verified by

$$\begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1$$

and

$$\begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1,$$

This verifies that $x_{14}(7)x_{14}(-7) = 1$ and $x_{14}(-7) = x_{14}(7) = 1$. □

MO14AS14. Calculate

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Solution. The inverse of the 6×6 matrix s_{25} is verified by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = 1,$$

This verifies that $s_{25}^2 = 1$. □

MO15HS21. Compute

$$\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 \end{pmatrix}.$$

Solution.

$$\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 79 & 64 & 94 & 89 \\ 37 & 48 & 31 & 47 \\ 5 & 97 & 26 & 77 \end{pmatrix}$$

□

MO16HS22. Compute

$$\frac{1}{3}\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}.$$

Solution.

$$\frac{1}{3}\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix} = \begin{pmatrix} 26 & \frac{62}{3} & 27 & \frac{85}{3} \\ 10\frac{2}{3} & \frac{41}{3} & 8 & 13 \\ 2 & 33 & \frac{29}{3} & 27 \end{pmatrix}$$

□

MO17HS23. Let

$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Compute $78E_{11} + 62E_{12} + 91E_{13} + 32E_{21} + 41E_{22} + 24E_{23}$.

Solution.

$$78E_{11} + 62E_{12} + 91E_{13} + 32E_{21} + 41E_{22} + 24E_{23} = \begin{pmatrix} 78 & 62 & 91 \\ 32 & 41 & 24 \end{pmatrix}$$

□

MO18HS24. Compute

$$\begin{pmatrix} 2 & 5 & 11 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 3 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}\begin{pmatrix} \frac{2}{100} \\ \frac{85}{100} \\ \frac{1}{100} \\ \frac{12}{100} \end{pmatrix}.$$

Solution.

$$\begin{pmatrix} 2 & 5 & 11 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 3 \\ -2 \end{pmatrix} = 2 \cdot 4 + 5 \cdot 0 + 11 \cdot 3 + 13 \cdot (-2) = 8 + 33 - 26 = 15$$

and

$$\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix} \begin{pmatrix} \frac{2}{100} \\ \frac{85}{100} \\ \frac{1}{100} \\ \frac{12}{100} \end{pmatrix} = \begin{pmatrix} \frac{78\cdot 2+62\cdot 85+91\cdot 1+85\cdot 12}{100} \\ \frac{32\cdot 2+41\cdot 85+24\cdot 1+39\cdot 12}{100} \\ \frac{6\cdot 2+99\cdot 85+29\cdot 1+81\cdot 12}{100} \end{pmatrix} = \begin{pmatrix} 65.37 \\ 40.41 \\ 94.28 \end{pmatrix}.$$

□

MO19HS25. Find $A \in M_2(\mathbb{Q})$ and $B \in M_2(\mathbb{Q})$ such that $AB \neq BA$.

Solution.

$$\text{If} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

then

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad BA = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

and so AB is *not the same* as BA. □

MO20AS15. Calculate the inner product $\langle (4,2,6,3)^t, (-1,2,0,-2i)^t \rangle$ using matrix multiplication.

Solution.

$$\begin{aligned} \langle (4,2,6,3)^t, (-1,2,0,-2i)^t \rangle &= \begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2i \end{pmatrix} \\ &= (4\cdot(-1) + 2\cdot 2 + 6\cdot 0 + 3\cdot(-2i)) = (-4+4+0-6i) = (-6i). \end{aligned}$$

□

1.2 Transpose and symmetric, Hermitian, orthogonal and unitary matrices

MO21HS61. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Compute A^T.

Solution. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$. □

MO22HS62. Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$. Prove that A is Hermitian and B is not Hermitian.

Solution. Since

$$A^* = \bar{A}^T = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} = A \quad \text{and} \quad B^* = \bar{B}^T = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \neq B$$

then A is Hermitian and B is not Hermitian. □

MO23HS63. Prove that the matrix $U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$ is unitary.

Solution. The matrix $U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$ is unitary since

$$UU^* = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

□

MO24HS64. Let $\theta \in \mathbb{R}$. Prove that $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

Solution. The matrix $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal since

$$\begin{aligned} QQ^T &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

□

1.3 Determinants and traces

MO25HS71. Let $A = \begin{pmatrix} 2 & 6 & 9 \\ 0 & 3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$. Find the determinant of A by factoring A.

Solution. Let $A = \begin{pmatrix} 2 & 6 & 9 \\ 0 & 3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$. Then

$$\begin{aligned} A &= h_1(2)h_2(3)h_3(-1) \begin{pmatrix} 1 & 3 & \frac{9}{2} \\ 0 & 1 & \frac{8}{3} \\ 0 & 0 & 1 \end{pmatrix} \\ &= h_1(2)h_2(3)h_3(-1)x_{12}(3)x_{13}(\tfrac{9}{2})x_{23}(\tfrac{8}{3}), \end{aligned}$$

So

$$\begin{aligned} \det(A) &= \det\big(h_1(2)h_2(3)h_3(-1)x_{12}(3))x_{13}(\tfrac{9}{2})x_{23}(\tfrac{8}{3}\big) \\ &= \det(h_1(2))\det(h_2(3))\det(h_3(-1)) \\ &\qquad \cdot \det(x_{12}(3)))\det(x_{13}(\tfrac{9}{2}))\det(x_{23}(\tfrac{8}{3})) \\ &= 2 \cdot 3 \cdot (-1) \cdot 1 \cdot 1 \cdot 1 = -6. \end{aligned}$$

□

MO26HS72. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the determinant of A by factoring A.

Solution. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Then

$$\begin{aligned} A &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{pmatrix} \\ &= s_1(-1)s_2(3) \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{pmatrix}. \end{aligned}$$

So $\det(A) = (-1)\cdot(-1)\cdot(-1)\cdot 1\cdot(-7) = 7$. □

MO27HS73. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Compute $\mathrm{tr}(A)$.

Solution.

$$\mathrm{tr}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1+5+9 = 15.$$

□

MO28HS74. Let $A, B \in M_{n\times n}(\mathbb{Q})$ and let $c \in \mathbb{Q}$. Prove that

$$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B), \qquad \mathrm{tr}(cA) = c\mathrm{tr}(A) \qquad \text{and} \qquad \mathrm{tr}(AB) = \mathrm{tr}(BA).$$

Solution.

$$\begin{aligned} \mathrm{tr}(A+B) &= \sum_{i=1}^n (A+B)(i,i) = \sum_{i=1}^n (A(i,i) + B(i,i)) \\ &= \Big(\sum_{i=1}^n A(i,i)\Big) + \Big(\sum_{i=1}^n B(i,i)\Big) = \mathrm{tr}(A) + \mathrm{tr}(B) \end{aligned}$$

and

$$\mathrm{tr}(cA) = \sum_{i=1}^n (cA)(i,i) = \sum_{i=1}^n (c\cdot A(i,i)) c \cdot \Big(\sum_{i=1}^n A(i,i)\Big) = c\mathrm{tr}(A)$$

and

$$\begin{aligned} \mathrm{tr}(AB) &= \sum_{i=1}^n (AB)(i,i) = \sum_{i=1}^n \sum_{j=1}^n A(i,j)B(j,i) = \sum_{i=1}^n \sum_{j=1}^n B(j,i)A(i,j) \\ &= \sum_{j=1}^n \sum_{i=1}^n B(j,i)A(i,j) = \sum_{j=1}^n (BA)(j,j) = \mathrm{tr}(BA). \end{aligned}$$

$\square$

MO29HS75. Let $n \in \mathbb{Z}_{>0}$. Let $A \in M_{n\times n}(\mathbb{Q})$ and let $P \in GL_n(\mathbb{Q})$ so that P is an invertible $n \times n$ matrix. Prove that

$$\det(PAP^{-1}) = \det(A) \qquad \text{and} \qquad \mathrm{tr}(PAP^{-1}) = \mathrm{tr}(A).$$

Solution. Let $A \in M_{n\times n}(\mathbb{Q})$. Since $1 = AA^{-1}$ then $\det(1) = \det(A)\det(A^{-1})$. So

$$1 = \det(A)\det(A^{-1}) \quad \text{and} \quad \frac{1}{\det(A)} = \det(A^{-1}).$$

Let $P \in GL_n(\mathbb{Q})$. Then

$$\begin{aligned}\det(PAP^{-1}) &= \det(P)\det(A)\det(P^{-1}) = \det(P)\det(A)\frac{1}{\det(P)} \\ &= \det(P)\frac{1}{\det(P)}\det(A) = \det(A)\end{aligned}$$

and

$$\mathrm{tr}(PAP^{-1}) = \mathrm{tr}((PA)P^{-1}) = \mathrm{tr}(P^{-1}(PA)) = \mathrm{tr}((P^{-1}P)A) = \mathrm{tr}(A).$$

$\square$

MO30HS76. Let $A \in M_{n\times n}(\mathbb{Q})$. Prove that

$$\frac{1}{\det(A)} = \det(A^{-1}).$$

Solution. Let $A \in M_{n\times n}(\mathbb{Q})$. Since $1 = AA^{-1}$ then $\det(1) = \det(A)\det(A^{-1})$. So

$$1 = \det(A)\det(A^{-1}) \quad \text{and} \quad \frac{1}{\det(A)} = \det(A^{-1}).$$

$\square$

MO31HS65. Assume $Q \in O_n(\mathbb{R})$. Using that $\det(AB) = \det(A)\det(B)$ and that $\det(A^T) = \det(A)$,

$$\text{prove that} \qquad \det(Q) \in \{1, -1\}.$$

Solution. Assume $Q \in O_n(\mathbb{R})$. Then $1 = QQ^T$ and

$$1 = \det(1) = \det(QQ^T) = \det(Q)\det(Q^T) = \det(Q)\det(Q) = \det(Q)^2.$$

So $\det(Q) \in \{1, -1\}$. $\square$

MO32HS163. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Compute the $(2, 3)$ cofactor of A.

(b) Use cofactor expansion along the third row to compute $\det(A)$.

Solution. Since $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ then the (2,3)-cofactor is

$$C_{23} = (-1)^{2+3} \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = (-1)^5 (1 \cdot 1 - 0 \cdot 2) = -1.$$

Using cofactor expansion along the third row,

$$\begin{aligned}
\det(A) &= (-1)^{3+1} \cdot 0 \cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} + (-1)^{3+2} \cdot 1 \cdot \det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} + (-1)^{3+3} \cdot 3 \cdot \det \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \\
&= 0 - (1 \cdot 1 - (-1) \cdot 1) + 3(1 \cdot 1 - (-1) \cdot 2) \\
&= 0 - 2 + 9 = 7.
\end{aligned}$$

□

MO33HS164. Using cofactor expansion down the fourth column, compute the determinant of

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 4 \end{pmatrix}.$$

Solution. Using cofactor expansion down the fourth column,

$$\begin{aligned}
\det(A) &= (-1)^{1+4} \cdot 1 \cdot \det \begin{pmatrix} 3 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & -4 & 2 \end{pmatrix} + 0 + 0 \\
&\qquad + (-1)^{4+4} \cdot 4 \cdot \det \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix} \\
&= -\Big((-1)^{1+1} \cdot 3 \cdot \det \begin{pmatrix} 0 & 1 \\ -4 & 2 \end{pmatrix} + (-1)^{2+1} \cdot 1 \cdot \det \begin{pmatrix} 2 & 2 \\ -4 & 2 \end{pmatrix} + 0\Big) \\
&\qquad + 4\Big((-1)^{1+1} \cdot 1 \cdot \det \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} + (-1)^{1+2} \cdot (-2) \cdot \det \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} + 0\Big) \\
&= -3(0+4) + (4+8) + 4\big((2-0) + 2(2-3)\big) \\
&= -12 + 12 + 8 + 8 = 16.
\end{aligned}$$

□