6 Working with linear transformations

6.1 Matrix of a linear transformation with respect to bases B and C

LT1HS98. Let $v = (1,5) \in \mathbb{R}^2$.

(a) Find the coordinate vector of v with respect to the basis $S = \{(1,0),(0,1)\}$.

(b) Find the coordinate vector of v with respect to the basis $B = \{(2,1),(-1,1)\}$.

Solution. (a) The coordinate vector of $v = (1,5)$ with respect to the basis $S = \{(1,0),(0,1)\}$ of $\mathbb{R}^2$ is

$$[v]_S = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \text{since} \quad (1,5) = 1\cdot(1,0) + 5\cdot(0.1).$$

(b) The coordinate vector of $v = (1,5)$ with respect to the basis $B = \{(2,1),(-1,1)\}$ of $\mathbb{R}^2$ is

$$[v]_B = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \text{since} \quad (1,5) = 2\cdot(2,1) + 3\cdot(-1.1).$$

□

LT2HS99. Find the coordinate vector of $p = 2+7x-9x^2$ with respect to the basis $B = \{2, \frac{1}{2}x, -3x^2\}$ of $\mathbb{Q}[x]_{\le 2}$.

Solution. The coordinate vector of $p = 2+7x-9x^2$ with respect to the basis $B = \{2, \frac{1}{2}x, -3x^2\}$ of $\mathbb{Q}[x]_{\le 2}$ is

$$[p]_B = \begin{pmatrix} 1 \\ 14 \\ 3 \end{pmatrix} \qquad \text{since} \quad 2+7x-9x^2 = 1\cdot 2 + 14\cdot(\tfrac{1}{2}x) + 3\cdot(-3x^2).$$

□

LT3HS100. The derivative with respect to x is the linear transformation $T\colon \mathbb{R}[x]_{\le 3} \to \mathbb{R}[x]_{\le 2}$ given by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$

(a) Find the matrix of T with respect to the basis $S = \{1, x, x^2, x^3\}$ of $\mathbb{R}[x]_{\le 3}$ and the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$.

(b) Find $\ker(T)$ and $\mathrm{im}(T)$.

Solution. Since

$$\begin{aligned}
T(1) &= T(1 + 0x + 0x^2 + 0x^3) = 0 + 0x + 0x^2, \\
T(x) &= T(0 + 1x + 0x^2 + 0x^3) = 1 + 0x + 0x^2, \\
T(x^2) &= T(0 + 0x + 1x^2 + 0x^3) = 0 + 2x + 0x^2, \\
T(x^3) &= T(0 + 0x + 0x^2 + 1x^3) = 0 + 0x + 3x^2,
\end{aligned}$$

then the matrix of T with respect to the basis $S = \{1, x, x^2, x^3\}$ of $\mathbb{R}[x]_{\le 3}$ and the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$ is

$$[T]_{BS} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Then

$$\begin{aligned}
\ker(T) &= \{p \in \mathbb{R}[x]_{\le 3} \mid T(p) = 0\} \\
&= \left\{ a_0 + a_1x + a_2x^2 + a_3x^3 \;\middle|\; \begin{array}{l} a_1 + 2a_2x + 3a_3x^2 \\ \quad = 0 + 0x + 0x^2 + 0x^3 \end{array} \right\} \\
&= \{a_0 + a_1x + a_2x^2 + a_3x^3 \mid a_1 = 0 \text{ and } a_2 = 0 \text{ and } a_3 = 0\} \\
&= \{a_0 + 0x + 0x^2 + 0x^3 \mid a_0 \in \mathbb{R}\} \\
&= \{a_0 \mid a_0 \in \mathbb{R}\} = \mathbb{R}\text{-span}\{1\}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{im}(T) &= \{T(p) \mid p \in \mathbb{R}[x]_{\le 3}\} \\
&= \{T(a_0 + a_1x + a_2x^2 + a_3x^3) \mid a_0, a_1, a_2, a_3 \in \mathbb{R}\} \\
&= \{a_1 + 2a_2x + 3a_3x^2 \mid a_1, a_2, a_3 \in \mathbb{R}\} = \mathbb{R}[x]_{\le 2},
\end{aligned}$$

□

LT4HS101. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function given by $T(x_1, x_2, x_3) = |x_2 - 2x_3, 3x_1 + x_3\rangle$.

(a) Prove that T is a linear transformation.

(b) Find the matrix of T with respect to the basis $S = \{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\}$ of $\mathbb{R}^3$ and the basis $B = \{|1,0\rangle, |0,1\rangle\}$ of $\mathbb{R}^2$.

Solution. FIX THIS UP Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function given by

$$T(x_1, x_2, x_3) = |x_2 - 2x_3, 3x_1 + x_3\rangle = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

With respect to the basis $S = \{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\}$ of $\mathbb{R}^3$ and the basis $B = \{|1,0\rangle, |0,1\rangle\}$ of $\mathbb{R}^2$ the matrix of T is

$$[T]_{BS} = \begin{pmatrix} 0 & 1 & -2 \\ 3 & 0 & 1 \end{pmatrix}.$$

□

LT5HS102. Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(Q) = Q^t.$$

Find the matrix of T with respect to the basis $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, where E_{ij} is the matrix with 1 in the (i,j) entry and 0 elsewhere.
Solution. Since

$$\begin{aligned}
T(E_{11}) &= E_{11} = 1 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 0 \cdot E_{22}, \\
T(E_{12}) &= E_{21} = 0 \cdot E_{11} + 0 \cdot E_{12} + 1 \cdot E_{21} + 0 \cdot E_{22}, \\
T(E_{21}) &= E_{12} = 0 \cdot E_{11} + 1 \cdot E_{12} + 0 \cdot E_{21} + 0 \cdot E_{22}, \\
T(E_{22}) &= E_{12} = 0 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 1 \cdot E_{22},
\end{aligned}$$

then the matrix of T with respect to the basis $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ is

$$[T]_{BB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

□

LT6HS103. Let $T\colon \mathbb{R}[x]_{\le 2} \to \mathbb{R}[x]_{\le 1}$ be the linear transformation given by

$$T(a_0 + a_1x + a_2x^2) = (a_0 + a_2) + a_0x.$$

(a) Find the matrix of T with respect to the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$ and the basis $C = \{1, x\}$ of $\mathbb{R}[x]_{\le 1}$.

(b) Find the matrix of T with respect to the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$ and the basis $D = \{2, 3x\}$ of $\mathbb{R}[x]_{\le 1}$.

Solution. Let $b_1 = 1$, $b_2 = x$, $b_3 = x^2$ and $c_1 = 2$, $c_2 = 3x$ and $d_1 = 1$, $d_2 = x$. Since

$$\begin{aligned} T(1) &= 1 + x & &= 1\cdot 1 + 1\cdot x & &= \tfrac{1}{2}\cdot 2 + \tfrac{1}{3}\cdot(3x), \\ T(x) &= 0 & &= 0\cdot 1 + 0\cdot x & &= 0\cdot 2 + 0\cdot(3x), \\ T(x^2) &= 1 & &= 1\cdot 1 + 0\cdot x & &= \tfrac{1}{2}\cdot 2 + 0\cdot(3x), \end{aligned}$$

then

$$\begin{aligned} T(b_1) &= 1\cdot d_1 + 1\cdot d_2, \\ T(b_2) &= 0\cdot d_1 + 0\cdot d_2, \\ T(b_3) &= 1\cdot d_1 + 0\cdot d_2, \end{aligned} \qquad \text{and} \qquad \begin{aligned} T(b_1) &= \tfrac{1}{2}c_1 + \tfrac{1}{3}c_2, \\ T(b_2) &= 0\cdot c_1 + 0\cdot c_2, \\ T(b_3) &= \tfrac{1}{2}\cdot c_1 + 0\cdot c_2, \end{aligned}$$

and

$$[T]_{DB} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad [T]_{CB} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \end{pmatrix}.$$

□

LT7HS104. Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and that the matrix of T with respect to
the basis $A = \{|1,0,0\rangle, |0,1,0\rangle, |0,0,1\rangle\}$ of $\mathbb{R}^3$ and
the basis $S = \{|1,0\rangle, |0,1\rangle\}$ of $\mathbb{R}^2$ is

$$[T]_{SA} = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix}.$$

Find the matrix of T with respect to
the basis $B = \{|1,1,0\rangle, |1,-1,0\rangle, |1,-1,-2\rangle\}$ of $\mathbb{R}^3$ and
the basis $C = \{|1,1\rangle, |1,-1\rangle\}$ of $\mathbb{R}^2$.

Solution. The answer is

$$[T]_{CB} = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 4 & 2 \end{pmatrix}$$

since

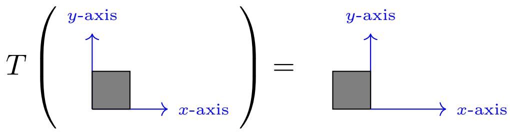
$$T(1,1,0) = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix} = 6 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$T(1,-1,0) = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 4 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$T(1,-1,-2) = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

$\square$

LT8HS105. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which is reflection across the y-axis.



Find the matrix of T with respect to the basis $S = \{|1,0\rangle, |0,1\rangle\}$.

Solution. Since

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} = (-1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and}$$
$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then the matrix of T with respect to the basis $S = \{|1,0\rangle, |0,1\rangle\}$ is

$$[T]_{SS} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

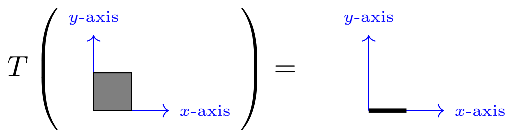
$\square$

LT9HS106. Find the matrix of the linear transformation

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{which is projection onto the } x \text{ axis.}$$

Is T injective? Is T surjective. Is T invertible?

Solution.



Since

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

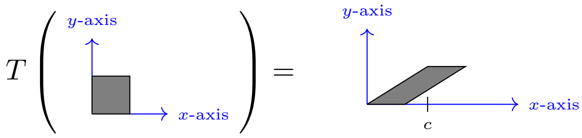
$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then the matrix of T with respect to the basis $S = \{|1,0\rangle, |0,1\rangle\}$ is

$$[T]_{SS} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The linear transformation T is not injective, not surjective, not invertible. $\square$

LT10HS107. Let $c \in \mathbb{R}_{>0}$ and let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is the shear by a factor of c along the x-axis.



Find the matrix of T with respect to the basis $S = \{|1,0\rangle, |0,1\rangle\}$.

Solution. Since

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ 1 \end{pmatrix} = c \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

then the matrix of T with respect to the basis $S = \{|1,0\rangle, |0,1\rangle\}$ is

$$[T]_{SS} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}.$$

The linear transformation T is invertible and

$$\ker(T) = \{0\} \quad \text{and} \quad \mathrm{im}(T) = \mathbb{R}^2.$$

$\square$

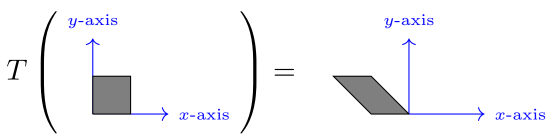
LT11HS108. Find the image of $(x, y) \in \mathbb{R}^2$ after a shear along the x-axis with $c = 1$ followed by a reflection across the y-axis.

Solution.

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x - y \\ y \end{pmatrix}.$$

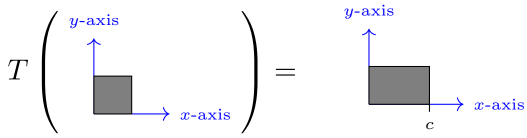
So

$$\begin{aligned} \mathrm{im}(T) &= \left\{ x\begin{pmatrix} -1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\} \\ &= \mathbb{R}\text{-span}\left\{ \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} = \mathbb{R}^2. \end{aligned}$$



$\square$

LT12HS109. Let $c \in \mathbb{R}_{>0}$ and let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is stretching of the x-axis by a factor of c.



Find the matrix of T with respect to the basis $B = \{(1,0),(0,1)\}$.

Solution. Since

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}c\\0\end{pmatrix} = c\cdot\begin{pmatrix}1\\0\end{pmatrix} + 0\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

and

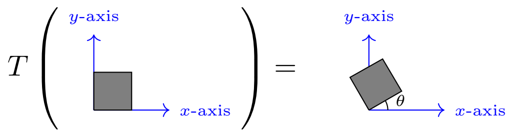
$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} = 0\cdot\begin{pmatrix}1\\0\end{pmatrix} + 1\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

then the matrix of T with respect to the basis $B = \{(1,0),(0,1)\}$ is

$$[T]_{BB} = \begin{pmatrix}c & 0\\ 0 & 1\end{pmatrix}.$$

$\square$

LT13HS110. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is rotation by θ (about the origin counterclockwise).



Find the matrix of T with respect to the basis $S = \{(1,0),(0,1)\}$.

Solution. Since

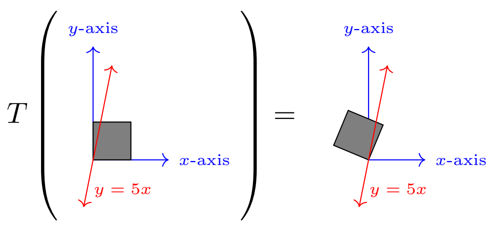
$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = \cos\theta\cdot\begin{pmatrix}1\\0\end{pmatrix} + \sin\theta\cdot\begin{pmatrix}0\\1\end{pmatrix} \qquad \text{and}$$
$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} = -\sin\theta\cdot\begin{pmatrix}1\\0\end{pmatrix} + \cos\theta\cdot\begin{pmatrix}0\\1\end{pmatrix}$$

then the matrix of T with respect to the basis $S = \{(1,0),(0,1)\}$ is

$$[T]_{SS} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \qquad \text{and} \qquad [T^{-1}]_{SS} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}$$

is the matrix of the rotation by $-\theta$ with respect to the basis S. $\square$

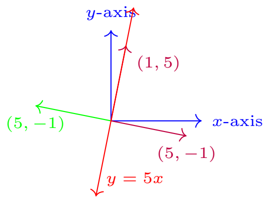
LT14HS111. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $y = 5x$.



Identify two lines through the origin that are invariant under T and find the image of the direction vectors for each of these lines.

Solution. Let

$$B = \{(1,5),(5,-1)\},$$



One line is the line $y = 5x$ and the other line is the line orthogonal to $y = 5x$. The line $y = 5x$ has slope 5 and the line orthogonal to $y = 5x$ has slope $-\frac{1}{5}$ and equation $y = -\frac{1}{5}x$. The corresponding direction vectors of these lines are $(1,5)$ and $(1,-\frac{1}{5})$ and

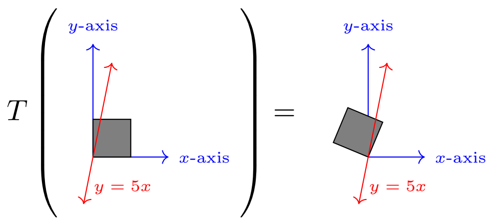
$$T(1,5) = (1,5) \qquad \text{and} \qquad T(1,-\tfrac{1}{5}) = -(1,-\tfrac{1}{5}) = (-1,\tfrac{1}{5}).$$

If $v_1 = (1,5)$ and $v_2 = (1,-\frac{1}{5})$ then

$$Tv_1 = 1\cdot v_1 \quad \text{and} \quad Tv_2 = (-1)\cdot v_2.$$

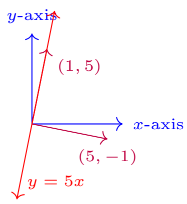
$\square$

LT15HS112. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is reflection in the line $y = 5x$.



Let $S = \{(1,0),(0,1)\}$ and let

$$B = \{(1,5),(5,-1)\},$$



so that the first vector in B is a vector in the direction of the line $y = 5x$ and the second vector in B is a vector perpendicular to $y = 5x$.

(a) Compute $[T]_{BB}$, $[I]_{SB}$, $[I]_{BS}$, and $[T]_{SS}$.

(b) Compute $T\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix}$.

Solution. Since

$$[T]_{BB} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad [I]_{SB} = \begin{pmatrix}1 & 5\\ 5 & -1\end{pmatrix}$$

and

$$[I]_{BS} = ([I]_{SB})^{-1} = -\tfrac{1}{26}\begin{pmatrix}-1 & -5\\ -5 & 1\end{pmatrix}$$

then the matrix of T with respect to the basis $S = \{(1,0),(0,1)\}$ is

$$[T]_{SS} = [I[_{SB}[T]_{BB}[I]_{BS} = -\tfrac{1}{26}\begin{pmatrix}1 & 5\\ 5 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-1 & -5\\ -5 & 1\end{pmatrix}$$
$$= -\tfrac{1}{26}\begin{pmatrix}1 & 5\\ 5 & -1\end{pmatrix}\begin{pmatrix}-1 & -5\\ 5 & -1\end{pmatrix} = -\tfrac{1}{26}\begin{pmatrix}24 & -10\\ -10 & -24\end{pmatrix} = \begin{pmatrix}-\frac{12}{13} & \frac{5}{13}\\ \frac{5}{13} & \frac{12}{13}\end{pmatrix}$$

In other words,

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-\frac{12}{13}\\ \frac{5}{13}\end{pmatrix} = -\frac{12}{13}\cdot\begin{pmatrix}1\\0\end{pmatrix} + \frac{5}{13}\cdot\begin{pmatrix}0\\1\end{pmatrix},$$

and

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\frac{5}{13}\\ \frac{12}{13}\end{pmatrix} = \frac{5}{13}\cdot\begin{pmatrix}1\\0\end{pmatrix} + \frac{12}{13}\cdot\begin{pmatrix}0\\1\end{pmatrix}.$$

□

LT16HS113. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x,y) = (3x - y, -x + 3y).$$

and let B and S be the bases of $\mathbb{R}^2$ given by

$$B = \{(1,1),(-1,1)\} \qquad \text{and} \qquad S = \{(1,0),(0,1)\}.$$

Let u and v be the vectors in $\mathbb{R}^2$ determined by

$$[u]_B = \begin{pmatrix}1\\1\end{pmatrix} \qquad \text{and} \qquad [v]_S = \begin{pmatrix}0\\2\end{pmatrix}.$$

(a) Compute $[I]_{BS}$, and $[I]_{SB}$

(b) Compute $[u]_S$ and $[v]_B$.

(c) Compute $[T]_{SS}$ and $[T]_{BB}$.

Solution. The change of basis matrices between B and S are

$$[I]_{BS} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} \qquad \text{and} \qquad [I]_{SB} = ([I]_{BS})^{-1} = \tfrac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$$

Then

$$[u]_S = \begin{pmatrix}0\\2\end{pmatrix} \qquad \text{and} \qquad [v]_B = \begin{pmatrix}1\\1\end{pmatrix}.$$

The matrices of T with respect to S and B are

$$[T]_{SS} = \begin{pmatrix}3 & -1\\ -1 & 3\end{pmatrix} \qquad \text{and} \qquad [T]_{BB} = \begin{pmatrix}2 & 0\\ 0 & 4\end{pmatrix}$$

so that T stretches by a factor of 2 in the direction $|1,1\rangle$ and T stretches by a factor of 4 in the direction $|1,-1\rangle$. □

6.2 Kernels and images of linear transformations

LT17HS96. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x,y,z) = (2x-y, y+z).$$

Find bases for $\ker(T)$ and $\mathrm{im}(T)$ and verify the rank-nullity theorem.

Solution.

$$\begin{aligned}
\ker(T) &= \{|x,y,z\rangle \in \mathbb{R}^3 \mid T(x,y,z) = |0,0\rangle\} \\
&= \{|x,y,z\rangle \in \mathbb{R}^3 \mid |2x-y, y+z\rangle = |0,0\rangle\} \\
&= \left\{ |x,y,z\rangle \in \mathbb{R}^3 \ \middle|\ \begin{matrix} 2x-y=0, \\ y+z=0 \end{matrix} \right\} \\
&= \left\{ |x,y,z\rangle \in \mathbb{R}^3 \ \middle|\ \begin{matrix} x = \frac{1}{2}y, \\ y=y, \\ z=-y \end{matrix} \right\} \\
&= \{y\cdot|\tfrac{1}{2},1,1\rangle \in \mathbb{R}^3 \mid y \in \mathbb{R}\} = \mathbb{R}\text{-span}\{|\tfrac{1}{2},1,-1\rangle\}
\end{aligned}$$

and $\{|\frac{1}{2},1,-1\rangle\}$ is a basis of $\ker(T)$. So $\dim(\ker(T)) = 1$.

Since

$$T(\tfrac{1}{2},0,0) = |1,0\rangle \quad \text{and} \quad T(0,0,1) = |0,1\rangle$$

then

$$|1,0\rangle \text{ and } |0,1\rangle \text{ are elements of } \mathrm{im}(T).$$

Since $\mathrm{im}(T)$ is a subspace of $\mathbb{R}^2$ then $\mathbb{R}\text{-span}\{|1,0\rangle,|0,1\rangle\}$ is a subset of $\mathrm{im}(T)$. So

$$\mathrm{im}(T) = \mathbb{R}^2 \quad \text{and} \quad \{|1,0\rangle,|0,1\rangle\} \text{ is a basis of } \mathrm{im}(T).$$

So $\dim(\mathrm{im}(T)) = 2$ and

$$\dim(\ker(T)) + \dim(\mathrm{im}(T)) = 2+1 = 3 \quad \text{and} \quad 3 = \dim(\mathbb{R}^3)$$

is the dimension of the source of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$. $\square$

LT18HS97. Let $T\colon \mathbb{R}[x]_{\le 2} \to \mathbb{R}[x]_{\le 1}$ be the linear transformation given by

$$T(a_0 + a_1x + a_2x^2) = (a_0 - a_1 + a_2)(1+2x).$$

(a) Find bases for $\ker(T)$ and $\mathrm{im}(T)$.
(b) Is T injective?
(c) Is T surjective?

Solution.

$$\begin{aligned}
\ker(T) &= \{a_0 + a_1x + a_2x^2 \in \mathbb{R}[x]_{\le 2} \mid T(a_0+a_1x+a_2x^2) = 0+0x\} \\
&= \{a_0 + a_1x + a_2x^2 \in \mathbb{R}[x]_{\le 2} \mid (a_0-a_1+a_2)(1+2x) = 0+0x\} \\
&= \left\{ a_0 + a_1x + a_2x^2 \in \mathbb{R}[x]_{\le 2} \ \middle|\ \begin{matrix} a_0 - a_1 + a_2 = 0, \\ 2(a_0 - a_1 + a_2) = 0 \end{matrix} \right\} \\
&= \{a_0 + a_1x + a_2x^2 \in \mathbb{R}[x]_{\le 2} \mid a_0 = a_1 - a_2\} \\
&= \{(a_1 - a_2) + a_1x + a_2x^2 \mid a_1, a_2 \in \mathbb{R}\} \\
&= \{a_1(1+x) + a_2(-1+x^2) \mid a_1,a_2 \in \mathbb{R}\} \\
&= \mathbb{R}\text{-span}\{1+x, -1+x^2\}
\end{aligned}$$

and $\{1+x, -1+x^2\}$ is a basis of $\ker(T)$.

$$\begin{aligned}
\mathrm{im}(T) &= \{T(a_0+a_1x+a_2x^2) \mid a_0,a_1,a_2\in\mathbb{R}\} \\
&= \{(a_0-a_1+a_2)(1+2x) \mid a_0,a_1,a_2\in\mathbb{R}\} \\
&= \{a(1+2x) \mid a\in\mathbb{R}\} = \mathbb{R}\text{-span}\{1+2x\}
\end{aligned}$$

and $\{1+2x\}$ is a basis of $\mathrm{im}(T)$. So $\dim(\mathrm{im}(T))=1$.

Since $\ker(T)\neq\{0\}$ then T is not injective.

Since $\mathbb{R}[x]_{\le 1} = \{c_0+c_1x \mid c_1,c_2\in\mathbb{R}\}$ then

$$\mathrm{im}(T)\neq\mathbb{R}[x]_{\le 1} \quad \text{and} \quad T \text{ is not surjective.}$$

$\square$