3 Kernels, images and solving linear equations

3.1 Solving systems of linear equations

KE1HS162. Find the values of $a, b \in \mathbb{Q}$ for which the system

$$\begin{array}{rcl} x - 2y + z = 4, & & \text{(a) no solution,} \\ 2x - 3y + z = 7, & \text{has} & \text{(b) a unique solution,} \\ 3x - 6y + az = b, & & \text{(c) LOTS of solutions.} \end{array}$$

Solution. In matrix form this system is

$$\begin{pmatrix} 3 & -6 & a \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 7 \\ 4 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 3 & -6 & a \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b \\ 4 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & a-3 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ b-12 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & a-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ b-12 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & a-3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ b-12 \end{pmatrix}.$$

Case 1: $a - 3 \neq 0$. Multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{a-3} \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ \frac{b-12}{a-3} \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 + \frac{b-12}{a-3} \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix}.$$

So

$$\begin{aligned} x &= 2 + \tfrac{b-12}{a-3}, \\ y &= -1 + \tfrac{b-12}{a-3}, \\ z &= \tfrac{b-12}{a-3}, \end{aligned}$$

or, equivalently,

$$\mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} 2 + \frac{b-12}{a-3} \\ -1 + \frac{b-12}{a-3} \\ \frac{b-12}{a-3} \end{pmatrix} \right\} \qquad \text{(exactly } \textit{one} \text{ solution).}$$

Case 2: $a - 3 = 0$. Then

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ b-12 \end{pmatrix}.$$

Case 2a: $b - 12 \neq 0$. If $b - 12 \neq 0$ then this system has *no solution.*

Case 2b: $b - 12 = 0$. If $b - 12 = 0$ then this system is

$$\begin{aligned} x - z &= 2, \\ y - z &= -1, \end{aligned} \qquad \text{which is} \qquad \begin{aligned} x &= 2 + z, \\ y &= -1 + z, \\ z &= 0 + z, \end{aligned}$$

where z can be any number. So

$$\mathrm{Sol}(Ax = b) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mathrm{span}\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} ,$$

and there are *LOTS of solutions.*

KE2HS39. (The number of solutions of a linear system)

(a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

(c) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

Solution. (a) If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then $\mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} 7 \\ 2 \end{pmatrix} \right\}$

$$\text{and} \qquad \begin{aligned} x_1 + 0x_2 &= 7, \\ 0x_1 + x_2 &= 2. \end{aligned} \qquad \text{has exactly one solution} \qquad \begin{aligned} x_1 &= 7, \\ x_2 &= 2. \end{aligned}$$

(b) If $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then $\mathrm{Sol}(Ax = b) = \emptyset$ and

$$\begin{aligned} x_1 + 0x_2 &= 7, \\ x_1 + 0x_2 &= 2. \end{aligned} \qquad \text{has no solutions.}$$

(c) If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ then $\mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} 7 \\ c \end{pmatrix} \;\middle|\; c \in \mathbb{Q} \right\}$,

$$\begin{aligned} x_1 + 0x_2 &= 7, \\ 0x_1 + 0x_2 &= 0. \end{aligned} \qquad \begin{array}{l} \text{has infinitely many} \\ \text{solutions} \end{array} \qquad \begin{aligned} x_1 &= 7, \\ x_2 &= c, \end{aligned} \quad \text{for any } c \in \mathbb{Q}.$$

□

KE3HS40. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x \\ y \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

Solution. If $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x \\ y \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ then $Ax = b$ is

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad \text{which is} \qquad \begin{aligned} 2x - y &= 3, \\ x + y &= 0. \end{aligned}$$

Start with

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{which is} \quad \begin{aligned} x &= 1, \\ y &= -1. \end{aligned}$$

$$\text{So} \qquad \mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \qquad \text{(exactly } \textit{one} \text{ solution).}$$

□

KE4HS41. Solve the following system of linear equations.

$$\begin{aligned} 4x - 2y + 5z &= 31, \\ 2x - 3y - 2z &= 13, \\ x - 3y + 2z &= 11. \end{aligned}$$

Solution. In matrix form, this is $Ax = b$, where

$$A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad b = \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix}.$$

Start with

$$\begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 4 & -2 & 5 \\ 1 & -3 & 2 \\ 0 & 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31 \\ 11 \\ -9 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -3 \\ 0 & 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -13 \\ -9 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{10}{3} \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -3 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \\ 17 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{17} \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \\ 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \\ 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix},$$

So

$$\begin{aligned} x &= 6, \\ y &= -1, \\ z &= 1, \end{aligned} \qquad \text{or, equivalently,} \quad \mathrm{Sol}(Ax=b) = \left\{ \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \right\}$$

(exactly *one* solution). □

KE5HS42. Let $A \in GL_n(\mathbb{Q})$ and let $x, b \in \mathbb{Q}^n$. Compute $\mathrm{Sol}(Ax = b)$.

Solution. Left multiplying both sides of $Ax = b$ by A^{-1} gives

$$A^{-1}Ax = A^{-1}b, \qquad \text{which says} \quad x = A^{-1}b.$$

So

$$\mathrm{Sol}(Ax = b) = \{A^{-1}b\}, \quad \text{which contains exactly one element.}$$

□

KE6HS43. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$.

(a) Compute $\mathrm{Sol}(Ax = b)$.

(b) Compute $\mathrm{Sol}(Ax = 0)$.

(c) Compute $\ker(A)$.

(d) Compute $\mathrm{im}(A)$.

Solution. If $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$ then $Ax = b$ is the system

$$\begin{aligned} x_1 + 0x_2 + x_3 &= -2, \\ 0x_1 + 2x_2 + 2x_3 &= 4, \\ 0x_1 + 0x_2 + 0x_3 &= -3, \end{aligned} \qquad \text{which has no solutions}$$

(no choice of $x_1, x_2 x_3 \in \mathbb{Q}$ will satisfy the third equation). So

$$\mathrm{Sol}(Ax = b) = \emptyset.$$

Then $Ax = 0$ is the system

$$\begin{aligned} x_1 + 0x_2 + x_3 &= 0, \\ 0x_1 + 2x_2 + 2x_3 &= 0, \\ 0x_1 + 0x_2 + 0x_3 &= 0, \end{aligned} \qquad \text{which is} \qquad \begin{aligned} x_1 &= -x_3, \\ x_2 &= x_3, \\ x_3 &= x_3, \end{aligned}$$

where x_3 can be any number. So

$$\mathrm{Sol}(Ax = 0) = \ker(A) = \left\{ x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{Q} \right\} = \mathbb{Q}\text{-span} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Then

$$\begin{aligned}
\mathrm{im}(A) &= \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ \Big|\ x_1, x_2, x_3 \in \mathbb{Q} \right\} \\
&= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \ \Big|\ x_1, x_2, x_3 \in \mathbb{Q} \right\} \\
&= \mathbb{Q}\text{-span} \left\{ \text{columns of } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \right\}
\end{aligned}$$

□

KE7HS44. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ 15 \end{pmatrix}$.

(a) Compute $\mathrm{Sol}(Ax = b)$.

(b) Compute $\mathrm{Sol}(Ax = 0)$.

(c) Compute $\ker(A)$.

(d) Compute $\mathrm{im}(A)$.

Solution. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ 15 \end{pmatrix}$ then

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 2, \\ 0x_1 + 1x_2 + 0x_3 &= -4, \\ 0x_1 + 0x_2 + x_3 &= 15, \end{aligned} \qquad \begin{matrix} \text{which has exactly} \\ \text{one solution} \end{matrix} \qquad \begin{aligned} x_1 &= 2, \\ x_2 &= -4, \\ x_3 &= 15. \end{aligned}$$

$$\mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} 2 \\ -4 \\ 15 \end{pmatrix} \right\}.$$

Then $Ax = 0$ is the system

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 0, \\ 0x_1 + 1x_2 + 0x_3 &= 0, \\ 0x_1 + 0x_2 + x_3 &= 0, \end{aligned} \qquad \begin{matrix} \text{which has exactly} \\ \text{one solution} \end{matrix} \qquad \begin{aligned} x_1 &= 0, \\ x_2 &= 0, \\ x_3 &= 0. \end{aligned}$$

So

$$\ker(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Then

$$\begin{aligned}
\mathrm{im}(A) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ \Big|\ x_1, x_2, x_3 \in \mathbb{Q} \right\} \\
&= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ \Big|\ x_1, x_2, x_3 \in \mathbb{Q} \right\} \\
&= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ \Big|\ x_1, x_2, x_3 \in \mathbb{Q} \right\} = \mathbb{Q}^3.
\end{aligned}$$

□

KE8HS45. Let $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(a) Compute $\mathrm{Sol}(Ax = b)$.

(b) Compute $\mathrm{Sol}(Ax = 0)$.

(c) Compute $\ker(A)$.

(d) Compute $\mathrm{im}(A)$.

Solution. (a) If

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{then}$$

$$\begin{array}{rl}
x_1 + 2x_2 + 0x_3 + 0x_4 + 5x_5 = 1, & \\
0x_1 + 0x_2 + 1x_3 + 0x_4 + 6x_5 = 2, & \text{which has an infinite} \\
0x_1 + 0x_2 + 0x_3 + x_4 + 7x_5 = 3, & \text{number of solutions.} \\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 0, &
\end{array}$$

More specifically,

$$\mathrm{Sol}(Ax = b) = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \ \Bigg|\ \begin{array}{l} x_1 = 1 - 2x_2 - 5x_5, \\ x_2 \in \mathbb{Q}, \\ x_3 = 2 - 6x_5, \\ x_4 = 3 - 7x_5, \\ x_5 \in \mathbb{Q} \end{array} \right\}$$

Equivalently,

$$\begin{aligned}
\mathrm{Sol}(Ax=b) &= \left\{ x = \begin{pmatrix}1\\0\\2\\3\\0\end{pmatrix} + \begin{pmatrix}-2x_2\\x_2\\0\\0\end{pmatrix} + \begin{pmatrix}-5x_5\\0\\-6x_5\\-7x_5\\x_5\end{pmatrix} \ \Big|\ x_2, x_5 \in \mathbb{Q} \right\} \\
&= \begin{pmatrix}1\\0\\2\\3\\0\end{pmatrix} + \left\{ x_2 \begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + x_5 \begin{pmatrix}-5\\0\\-6\\-7\\1\end{pmatrix} \ \Big|\ x_2, x_5 \in \mathbb{Q} \right\} \\
&= \begin{pmatrix}1\\0\\2\\3\\0\end{pmatrix} + \mathbb{Q}\text{-span}\left\{ \begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-5\\0\\-6\\-7\\1\end{pmatrix} \right\} \\
&= p + \ker(A).
\end{aligned}$$

(b) The system of equations $Ax = 0$ is

$$\begin{aligned}
x_1 + 2x_2 + 0x_3 + 0x_4 + 5x_5 &= 0, \\
0x_1 + 0x_2 + 1x_3 + 0x_4 + 6x_5 &= 0, \\
0x_1 + 0x_2 + 0x_3 + x_4 + 7x_5 &= 0, \\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0,
\end{aligned} \qquad \begin{matrix}\text{which has an infinite}\\ \text{number of solutions.}\end{matrix}$$

More specifically,

$$\mathrm{Sol}(Ax=0) = \left\{ x = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} \ \Bigg|\ \begin{matrix} x_1 = 0 - 2x_2 - 5x_5, \\ x_2 \in \mathbb{Q}, \\ x_3 = 0 - 6x_5, \\ x_4 = 0 - 7x_5, \\ x_5 \in \mathbb{Q} \end{matrix} \right\}$$

Equivalently,

$$\begin{aligned}
\mathrm{Sol}(Ax=0) &= \left\{ x = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}-2x_2\\x_2\\0\\0\end{pmatrix} + \begin{pmatrix}-5x_5\\0\\-6x_5\\-7x_5\\x_5\end{pmatrix} \ \Big|\ x_2, x_5 \in \mathbb{Q} \right\} \\
&= \left\{ x_2 \begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + x_5 \begin{pmatrix}-5\\0\\-6\\-7\\1\end{pmatrix} \ \Big|\ x_2, x_5 \in \mathbb{Q} \right\} \\
&= \mathbb{Q}\text{-span}\left\{ \begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-5\\0\\-6\\-7\\1\end{pmatrix} \right\} = \ker(A).
\end{aligned}$$

(c) Since $\ker(A) = \mathrm{Sol}(Ax = 0)$ then $\ker(A)$ is determined in (b).
(d)

$$\begin{aligned}
\mathrm{im}(A) &= \left\{ \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \ \Big|\ x_1, x_2, x_3, x_4, x_5 \in \mathbb{Q} \right\} \\
&= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 5 \\ 6 \\ 7 \\ 0 \end{pmatrix} \ \Big|\ x_1, x_2, x_3, x_4, x_5 \in \mathbb{Q} \right\} \\
&= \mathbb{Q}\text{-span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 7 \\ 0 \end{pmatrix} \right\}.
\end{aligned}$$

Since

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 5 \\ 6 \\ 7 \\ 0 \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 6 \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 7 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

then

$$\mathrm{im}(A) = \mathbb{Q}\text{-span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$\square$

KE9HS46. Let $S = \{|1, 3, -1, 1\rangle,\ |2, 6, 0, 4\rangle,\ |3, 9, -2, 4\rangle\ \}$. Show that

$$\mathbb{R}\text{-span}(S) = \mathrm{im}(A), \text{ where } A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}.$$

Solution.

$$\begin{aligned}
\mathrm{im}(A) &= \left\{ \begin{pmatrix} 1 & 3 & -1 & 1 \\ 2 & 6 & 0 & 4 \\ 3 & 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ \Big|\ x_1, x_2, x_3 \in \mathbb{R} \right\} \\
&= \left\{ x_1 \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 6 \\ 0 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} 3 \\ 0 \\ 2 \\ 4 \end{pmatrix} \ \Big|\ x_1, x_2, x_3, x_4, x_5 \in \mathbb{R} \right\} \\
&= \mathbb{R}\text{-span} \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \\ 4 \end{pmatrix} \right\} \\
&= \mathbb{R}\text{-span}(S).
\end{aligned}$$

$\square$

KE10HS47. (How kernels and images change) Let $A \in M_{t\times s}(\mathbb{Q})$ and $P \in GL_t(\mathbb{Q})$ and $Q \in GL_s(\mathbb{Q})$. Prove that

$$\ker(PA) = \ker(A), \qquad \ker(AQ) = Q^{-1}\cdot\ker(A),$$
$$\mathrm{im}(PA) = P\cdot\mathrm{im}(A), \qquad \mathrm{im}(AQ) = \mathrm{im}(A).$$

Solution.

$$\begin{aligned}
\ker(PA) &= \{x \in \mathbb{Q}^s \mid PAx = 0\} = \{x \in \mathbb{Q}^s \mid P^{-1}PAx = P^{-1}0\} \\
&= \{x \in \mathbb{Q}^s \mid Ax = 0\} = \ker(A), \\
\ker(AQ) &= \{x \in \mathbb{Q}^s \mid AQx = 0\} = \{Q^{-1}Qx \in \mathbb{Q}^s \mid AQx = 0\} \\
&= Q^{-1}\cdot\{Qx \in \mathbb{Q}^s \mid AQx = 0\} \\
&= Q^{-1}\{y \in \mathbb{Q}^s \mid Ay = 0\} = Q^{-1}\cdot\ker(A), \\
\mathrm{im}(PA) &= \{PAx \mid x \in \mathbb{Q}^s\} = P\{Ax \mid x \in \mathbb{Q}^s\} = P\cdot\mathrm{im}(A), \\
\mathrm{im}(AQ) &= \{AQx \mid x \in \mathbb{Q}^s\} = \{Ay \mid Q^{-1}y \in \mathbb{Q}^s\} \\
&= \{Ay \mid y \in \mathbb{Q}^s\} = \mathrm{im}(A).
\end{aligned}$$

$\square$

KE11HS48. Let $A \in M_{t\times s}(\mathbb{Q})$. Show that $\ker(A)$ is a subspace of $\mathbb{Q}^s$.

Solution. (a) Since $A0 = 0$ then $0 \in \ker(A)$.

(b) Assume $w_1, w_2 \in \ker(A)$. Then $Aw_1 = 0$ and $Aw_2 = 0$. So

$$A(w_1 + w_2) = Aw_1 + Aw_2 = 0 + 0 = 0. \quad \text{So } w_1 + w_2 \in \ker(A).$$

(c) Assume $w \in \ker(A)$ and $c \in \mathbb{Q}$. Then $Aw = 0$ and

$$A(cw) = cAw = c0 = 0. \quad \text{So } cw \in \ker(A).$$

So $\ker(A)$ is a subspace of $\mathbb{Q}^s$. $\square$

KE12HS49. Let $A \in M_{t\times s}(\mathbb{Q})$. Show that $\mathrm{im}(A)$ is a subspace of $\mathbb{Q}^t$.

Solution. (a) Since $0 = A0$ then $0 \in \mathrm{im}(A)$.

(b)Assume $y_1, y_2 \in \mathrm{im}(A)$. Then there exist $x_1, x_2 \in \mathbb{Q}^s$ such that $y_1 = Ax_1$ and $y_2 = Ax_2$. Then

$$y_1 + y_2 = Ax_1 + Ax_2 = A(x_1 + x_2). \quad \text{So } y_1 + y_2 \in \mathrm{im}(A).$$

(c) Assume $y \in \mathrm{im}(A)$ and $c \in \mathbb{Q}$. Then there exists $x \in \mathbb{Q}^s$ such that $y = Ax$. Then

$$cy = cAx = A(cx). \quad \text{So } cy \in \mathrm{im}(A).$$

So $\mathrm{im}(A)$ is a subspace of $\mathbb{Q}^t$. $\square$

KE13HS54. Let

$$1_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis of $\ker(1_2)$.

(b) Find a basis of $\mathrm{im}(1_2)$.

Solution. Let $\{l_1, \ldots, l_5\}$ be the standard basis of $\mathbb{Q}^5$. Then

$$\ker(1_2) = \mathrm{span}\left\{\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}\right\} = \mathrm{span}\{l_3, l_4, l_5\} \quad \text{and } \{l_3, l_4, l_5\} \text{ is a basis of } \ker(1_2).$$

Let $\{r_1, \ldots, r_6\}$ be the standard basis of $\mathbb{Q}^6$. Then

$$\mathrm{im}(1_2) = \mathrm{span}\left\{\begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}\right\} = \mathrm{span}\{r_1, r_2\} \quad \text{and } \{r_1, r_2\} \text{ is a basis of } \mathrm{im}(1_2).$$

□

KE14HS50. Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q}).$$

Let

$$A = P1_rQ.$$

Prove that

$$\ker(A) \text{ has basis } \{\text{last } s{-}r \text{ columns of } Q^{-1}\},$$
$$\mathrm{im}(A) \text{ has basis } \{\text{first } r \text{ columns of } P\}.$$

Solution. By the How Kernel and Image Change Proposition

$$\begin{aligned}\ker(A) &= \ker(P1_rQ) = \ker(1_rQ) = Q^{-1}\ker(1_r) \quad \text{and}\\ \mathrm{im}(A) &= \mathrm{im}(P1_rQ) = \mathrm{im}(P1_r) = P\mathrm{im}(1_r)\end{aligned}$$

Since $\{Q^{-1}e_{r+1}, \ldots, Q^{-1}e_s\}$ is a basis of $Q^{-1}\ker(1_r)$ then

$$\ker(A) \text{ has basis } \{\text{last } s{-}r \text{ columns of } Q^{-1}\},$$

Since $\{Pe_1, \ldots, Pe_r\}$ is a basis of $P\mathrm{im}(1_r)$ then

$$\mathrm{im}(A) \text{ has basis } \{\text{first } r \text{ columns of } P\}. \qquad \square$$

KE15HS51. Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q}).$$

Let

$$A = P1_rQ.$$

Prove that

$$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = (\text{number of columns of } A).$$

Solution. Let $A \in M_{t\times s}(\mathbb{Q})$. By definition,

$$\begin{gathered}\dim(\ker(A)) \text{ is the number of elements in a basis of } \ker(A), \text{ and} \\ \dim(\mathrm{im}(A)) \text{ is the number of elements in a basis of } \mathrm{im}(A).\end{gathered}$$

From the Computing Kernels and Images Theorem,

$$\dim(\ker(A)) = s - r \quad \text{and} \quad \dim(\mathrm{im}(A)) = r.$$

Since $(s-r)+r = s$ then

$$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = (\text{number of columns of } A).$$

□

KE16HS52. Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. Let $r \in \{1,\ldots,\min(s,t)\}$ and let

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q}).$$

Let

$$A = P1_rQ.$$

Prove that $\dim(\mathrm{im}(A)) = \mathrm{rank}(A)$.

Solution. Let $A \in M_{t\times s}(\mathbb{Q})$. By definition,

$$\begin{gathered}\dim(\ker(A)) \text{ is the number of elements in a basis of } \ker(A), \text{ and} \\ \dim(\mathrm{im}(A)) \text{ is the number of elements in a basis of } \mathrm{im}(A).\end{gathered}$$

From the Computing Kernels and Images Theorem,

$$\dim(\ker(A)) = s - r \quad \text{and} \quad \dim(\mathrm{im}(A)) = r = \mathrm{rank}(A).$$

□

KE17HS55. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}$.

(a) Show that $A = P1_2Q$, where

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Compute Q^{-1}.

(c) Find a basis of $\ker(A)$.

(d) Find a basis of $\mathrm{im}(A)$.

Solution. By multiplying out

$$P1_2Q = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 6 & 0 \\ -1 & 0 & 1 \\ 1 & 4 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix} = A.$$

(b) Since

$$Q \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$Q^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Since

$$\ker(A) = \ker(P1_2Q) = \ker(1_2Q) = Q^{-1}\ker(1_2)$$

$$\text{and} \quad Q^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{then} \quad \ker(A) = \mathbb{Q}\text{-span}\left\{ \begin{pmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{pmatrix} \right\}.$$

Thus $\ker(A)$ has basis $\{|-2, -\frac{1}{2}, 1\rangle\}$.

(d) Since

$$\mathrm{im}(A) = \mathrm{im}(P1_2Q) = P\mathrm{im}(1_2) = \mathrm{span}\left\{ P\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, P\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

then

$$\mathrm{im}(A) \quad \text{has basis} \left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 0 \\ 4 \end{pmatrix} \right\}.$$

□

KE18AQ3. Let

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

and let rowspace(M) be the set of linear combinations of the rows of M. Find a basis of rowspace(M).

Solution. The matrix multiplication

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

checks that

$$M = PR, \qquad \text{where} \qquad P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $M = PR$ and $R = P^{-1}M$ then

(a) The rows of R are linear combinations of the rows of M and
(b) The rows of M are linear combinations of the rows of R.

Thus

$$\mathrm{rowspace}(M) = \mathrm{rowspace}(R).$$

Since $\{(1\ 0\ 0\ -1\ 0),\ (0\ 1\ 0\ \frac{1}{2}\ \frac{1}{2}),\ (0\ 0\ 1\ \frac{1}{2}\ \frac{1}{2})\}$ is a basis of rowspace(R) then

$$\{(1\ 0\ 0\ -1\ 0),\ (0\ 1\ 0\ \tfrac{1}{2}\ \tfrac{1}{2}),\ (0\ 0\ 1\ \tfrac{1}{2}\ \tfrac{1}{2})\} \quad \text{is a basis of rowspace}(M).$$

□

Ke19AQ4. Let

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

and let colspace(M) be the set of linear combinations of the columns of M. Find a basis of colspace(M).

Solution. The matrix multiplication

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

checks that

$$M = PR, \qquad \text{where} \qquad P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $M = PR$ then colspace$(M) = \{P\vec{c} \mid \vec{c} \in \mathrm{colspace}(R)\}$. Since colspace$(R)$ has basis consisting of the first three columns of R and $PR = M$ then

$$\mathrm{colspace}(M) \quad \text{has basis} \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad \text{(the first three columns of } M\text{).}$$

□

KE20AQ5. Let
$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$
Find a basis of $\ker(M)$.

Solution. The matrix multiplication
$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
checks that
$$M = PR, \qquad \text{where} \qquad P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 4 & 0 \\ 1 & 4 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad R = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
The *kernel* of M is the set
$$\ker(M) = \left\{ \vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \;\middle|\; M\vec{c} = 0 \right\}$$
Since
$$\ker(M) = \ker(PR) = \ker(R) \qquad \text{is the set of solutions to the system} \qquad \begin{aligned} c_1 - c_4 &= 0, \\ c_2 + \tfrac{1}{2}c_4 + \tfrac{1}{2}c_5 &= 0, \\ c_3 + \tfrac{1}{2}c_4 + \tfrac{1}{2}c_5 &= 0, \end{aligned}$$
then
$$\ker(M) = \left\{ c_4 \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + c_5 \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \;\middle|\; c_4, c_5 \in \mathbb{R} \right\} \qquad \text{which has basis} \qquad \left\{ \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \right\}.$$

□