2 Inverses, normal form and rank

2.1 Inverses

IF1HS26. Let $c \in \mathbb{Q}$. Find the inverse of the matrix $\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$.

Solution. Since

$$\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

then $\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -c \end{pmatrix}$. □

IF2HS27. Let $c \in \mathbb{Q}$. FInd the inverses of the matrices

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

Solution.

$$x_{12}(c)^{-1} = \begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{13}(c)^{-1} = \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_{23}(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix},$$

since

$$x_{12}(c)x_{12}(c)^{-1} = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + c\cdot 0 & 1\cdot(-c) + c\cdot 1 & 0 \\ 0\cdot 1 + 1\cdot 0 & 0\cdot(-c) + 0\cdot 1 & 0 \\ 0 & 0 & 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$x_{12}(c)^{-1}x_{12}(c) = \begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + (-c)\cdot 0 & 1\cdot c + (-c)\cdot 1 & 0 \\ 0\cdot 1 + 1\cdot 0 & 0\cdot c + 0\cdot 1 & 0 \\ 0 & 0 & 1\cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and similarly for $x_{13}(c)$ and $x_{23}(c)$. □

IF3HS28. Let $d \in \mathbb{Q}$ and assume $d \neq 0$. Find the inverses of the matrices

$$h_1(d) = \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; h_2(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; h_3(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

Solution.

$$h_1(d)^{-1} = \begin{pmatrix} \frac{1}{d} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad h_2(d)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{d} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad h_3(d)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{d} \end{pmatrix},$$

since

$$h_1(d)h_1(d)^{-1} = \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{d} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} d \cdot \frac{1}{d} & 0 & 0 \\ 0 & 1 \cdot 1 & 0 \\ 0 & 0 & 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$h_1(d)^{-1}h_1(d) = \begin{pmatrix} \frac{1}{d} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{d} \cdot d & 0 & 0 \\ 0 & 1 \cdot 1 & 0 \\ 0 & 0 & 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and similarly for $h_2(d)$ and $h_3(d)$. $\square$

IF4HS29. Let $c \in \mathbb{Q}$. Find the inverses of the matrices

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Solution.

$$s_1(c)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -c & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c \end{pmatrix},$$

since

$$s_1(c)s_1(c)^{-1} = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -c & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c \cdot 0 + 1 \cdot 1 & c \cdot 1 + 1 \cdot (-c) & 0 \\ 1 \cdot 0 + 0 \cdot 1 & c \cdot 1 + 0 \cdot (-c) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$s_1(c)^{-1}s_1(c) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot c + 1 \cdot 1 & 0 \cdot 1 + 1 \cdot 0 & 0 \\ 1 \cdot c + (-c) \cdot 1 & 1 \cdot 1 + (-c) \cdot 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and similarly for $s_2(c)$. $\square$

IF5HS30. Find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

Solution. Start with $AA^{-1} = 1$ which is

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Left multiply both sides by $s_1(-1)^{-1}$, which is the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Left multiply both sides by $s_2(1)^{-1}$, which is the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

Left multiply both sides by $h_3(-1)^{-1}h_1(-1)^{-1}$, which is the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Left multiply both sides by $x_{23}(3)^{-1}$, which is the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Left multiply both sides by $x_{13}(-1)^{-1}$, which is the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} -1 & -2 & 1 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Left multiply both sides by $x_{12}(1)^{-1}$, which is the matrix

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In summary,

$$A^{-1} = x_{12}(1)^{-1}x_{13}(-1)^{-1}x_{23}(3)^{-1} \cdot h_3(-1)^{-1}h_1(-1)^{-1} \cdot s_2(1)^{-1}s_1(-1)^{-1}$$

and

$$A = s_1(-1)s_2(1) \cdot h_1(-1)h_3(-1) \cdot x_{23}(3)x_{13}(-1)x_{12}(1).$$

$\square$

IF6HS31. Find the inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

Solution. Start with $AA^{-1} = 1$ which is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 3 & 4 \\ 0 & \frac{2}{3} \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{3}{2} & -2 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

□

IF7HS32. Let $A, B \in GL_n(\mathbb{Q})$. Prove that $AB \in GL_n(\mathbb{Q})$ by finding $(AB)^{-1}$.

Solution. By associativity,

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1} \cdot 1 \cdot B = B^{-1}B = 1,$$

and

$$(AB)(B^{-1}A^{-1}) = A^{-1}(B^{-1}B)A = A^{-1} \cdot 1 \cdot A = A^{-1}A = 1.$$

So $(AB)^{-1} = B^{-1}A^{-1}$. □

2.2 Normal form

IF8HS33. By multiplying out the matrices on the left hand side check that

$$s_4(1)s_3(2)s_2(3)s_1(4) \cdot s_4(5)s_3(6)s_2(7) \cdot s_4(8)s_3(9) \cdot s_4(10) = \begin{pmatrix} 4 & 7 & 9 & 10 & 1 \\ 3 & 6 & 8 & 1 & 0 \\ 2 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution.

$$\begin{aligned}
\begin{pmatrix} 4 & 7 & 9 & 10 & 1 \\ 3 & 6 & 8 & 1 & 0 \\ 2 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}
&= s_4(1) \begin{pmatrix} 4 & 7 & 9 & 10 & 1 \\ 3 & 6 & 8 & 1 & 0 \\ 2 & 5 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
&= s_4(1)s_3(2) \begin{pmatrix} 4 & 7 & 9 & 10 & 1 \\ 3 & 6 & 8 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
&= s_4(1)s_3(2)s_2(3) \begin{pmatrix} 4 & 7 & 9 & 10 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 6 & 8 & 1 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \\
&= s_4(1)s_3(2)s_2(3)s_1(4) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 7 & 9 & 10 & 1 \\ 0 & 6 & 8 & 1 & 0 \\ 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

and

$$\begin{pmatrix}4&7&9&10&1\\3&6&8&1&0\\2&5&1&0&0\\1&1&0&0&0\\1&0&0&0&0\end{pmatrix} = s_4(1)s_3(2)s_2(3)s_1(4)s_4(5)\begin{pmatrix}1&0&0&0&0\\0&7&9&10&1\\0&6&8&1&0\\0&1&0&0&0\\0&0&1&0&0\end{pmatrix}$$
$$= s_4(1)s_3(2)s_2(3)s_1(4)s_4(5)s_3(6)\begin{pmatrix}1&0&0&0&0\\0&7&9&10&1\\0&1&0&0&0\\0&0&8&1&0\\0&0&1&0&0\end{pmatrix}$$
$$= s_4(1)s_3(2)s_2(3)s_1(4)s_4(5)s_3(6)s_2(7)\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&9&10&1\\0&0&8&1&0\\0&0&1&0&0\end{pmatrix}$$

and

$$\begin{pmatrix}4&7&9&10&1\\3&6&8&1&0\\2&5&1&0&0\\1&1&0&0&0\\1&0&0&0&0\end{pmatrix} = s_4(1)s_3(2)s_2(3)s_1(4)s_4(5)s_3(6)s_2(7)s_4(8)\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&9&10&1\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}$$
$$= s_4(1)s_3(2)s_2(3)s_1(4)s_4(5)s_3(6)s_2(7)s_4(8)s_3(9)\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&10&1\\0&0&0&1&0\end{pmatrix}$$

and

$$\begin{pmatrix}4&7&9&10&1\\3&6&8&1&0\\2&5&1&0&0\\1&1&0&0&0\\1&0&0&0&0\end{pmatrix} = s_4(1)s_3(2)s_2(3)s_1(4)s_4(5)s_3(6)s_2(7)s_4(8)s_3(9)s_4(10)\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

$\square$

IF9HS34. By multiplying out the matrices on the left hand side check that

$$h_1(15)h_2(-3)h_3(76)h_4(-19)h_5(2) = \begin{pmatrix}15&0&0&0&0\\0&-3&0&0&0\\0&0&76&0&0\\0&0&0&-19&0\\0&0&0&0&2\end{pmatrix}.$$

Solution.

$$\begin{aligned}
\begin{pmatrix} 15 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 76 & 0 & 0 \\ 0 & 0 & 0 & -19 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}
&= h_1(15) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 76 & 0 & 0 \\ 0 & 0 & 0 & -19 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \\
&= h_1(15)h_2(-3) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 76 & 0 & 0 \\ 0 & 0 & 0 & -19 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \\
&= h_1(15)h_2(-3)h_3(76) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -19 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \\
&= h_1(15)h_2(-3)h_3(76)h_4(-19) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \\
&= h_1(15)h_2(-3)h_3(76)h_4(-19)h_5(2) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
&= h_1(15)h_2(-3)h_3(76)h_4(-19)h_5(2).
\end{aligned}$$

□

IF10HS35. (a) Check that

$$\begin{pmatrix} 1 & -10 & 71 & -302 & 186 \\ 0 & 1 & -8 & 34 & -21 \\ 0 & 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 10 & 9 & 7 & 4 \\ 0 & 1 & 8 & 6 & 3 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) By multiplying out the matrices on the left hand side check that

$$\begin{matrix} x_{45}(1)x_{35}(2)x_{25}(3) \\ \cdot x_{15}(4)x_{34}(5)x_{24}(6) \\ \cdot x_{14}(7)x_{23}(8)x_{13}(9) \\ \cdot x_{12}(10) \end{matrix} = \begin{pmatrix} 1 & 10 & 9 & 7 & 4 \\ 0 & 1 & 8 & 6 & 3 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) By multiplying out the matrices on the left hand side check that

$$\begin{matrix} x_{12}(10)^{-1}x_{13}(9)^{-1}x_{23}(8)^{-1} \\ \cdot x_{14}(7)^{-1}x_{24}(6)^{-1}x_{34}(5)^{-1} \\ \cdot x_{15}(4)^{-1}x_{25}(3)^{-1}x_{35}(2)^{-1} \\ \cdot x_{45}(1)^{-1} \end{matrix} = \begin{pmatrix} 1 & -10 & 71 & -302 & 186 \\ 0 & 1 & -8 & 34 & -21 \\ 0 & 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution. FILL THIS OUT □

IF11HS36. (a) Check that

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) By multiplying out the matrices on the left hand side check that

$$s_1(\tfrac{1}{3})h_1(3)h_2(\tfrac{2}{3})x_{12}(\tfrac{4}{3}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(c) By multiplying out the matrices on the left hand side check that

$$x_{12}(\tfrac{4}{3})^{-1}h_2(\tfrac{2}{3})^{-1}h_1(3)^{-1}s_1(\tfrac{1}{3})^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

Solution. (a)

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} (-2)\cdot 1 + 1\cdot 3 & (-2)\cdot 2 + 1\cdot 4 \\ \frac{3}{2}\cdot 1 + (-\frac{1}{2})\cdot 3 & \frac{3}{2}\cdot 2 + (-\frac{1}{2})\cdot 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b)

$$\begin{aligned} s_1(\tfrac{1}{3})h_1(3)h_2(\tfrac{2}{3})x_{12}(\tfrac{4}{3}) &= \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 0 & \frac{2}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3}\cdot 3 + 1\cdot 0 & \frac{1}{3}\cdot 4 + \frac{2}{3} \\ 1\cdot 3 + 0\cdot 0 & 4\cdot 1 + 0\cdot \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}. \end{aligned}$$

(c)

$$\begin{aligned} x_{12}(\tfrac{4}{3})^{-1}h_2(\tfrac{2}{3})^{-1}h_1(3)^{-1}s_1(\tfrac{1}{3})^{-1} &= \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -2 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3}\cdot 0 + (-2)\cdot 1 & \frac{1}{3}\cdot 1 + (-2)\cdot(-\frac{1}{3}) \\ 0\cdot 0 + \frac{3}{2}\cdot 1 & 0\cdot 1 + \frac{3}{2}\cdot(-\frac{1}{3}) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}. \end{aligned}$$

□

IF12HS37. (a) Check that

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) By multiplying out the matrices on the left hand side check that

$$s_1(-1)s_2(1)h_1(-1)h_3(-1)x_{23}(3)x_{13}(-1)x_{12}(1) = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

(c) By multiplying out the matrices on the left hand side check that

$$\begin{matrix} x_{12}(1)^{-1}x_{13}(-1)^{-1}x_{23}(3)^{-1} \\ \cdot h_3(-1)^{-1}h_1(-1)^{-1}s_2(1)^{-1}s_1(-1)^{-1} \end{matrix} = \begin{pmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}.$$

Solution. FILL THIS OUT $\square$

IF13HS38. Let $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{pmatrix} \in M_{3\times 4}(\mathbb{Q})$.
By multiplying out the matrices on the left hand side check that

$$s_2(0)s_1(1)s_2(2)x_{12}(-3)\cdot 1_2 \cdot x_{23}(1)x_{14}(5)x_{24}(-2) = A.$$

Determine $\mathrm{rank}(A)$.

Solution. Let $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{pmatrix} \in M_{3\times 4}(\mathbb{Q})$. Then

$$\begin{aligned} A &= s_2(0)\begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & -2 \end{pmatrix} = s_2(0)s_1(1)\begin{pmatrix} 1 & -3 & 0 & 5 \\ 0 & 2 & 2 & -4 \\ 0 & 1 & 1 & -2 \end{pmatrix} \\ &= s_2(0)s_1(1)s_2(2)\begin{pmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &= s_2(0)s_1(1)s_2(2)x_{12}(-3)\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

Since this last right hand factor has a row of 0s then A is not invertible. Then

$$\begin{aligned} A &= s_2(0)s_1(1)s_2(2)x_{12}(-3)\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}x_{24}(-2) \\ &= s_2(0)s_1(1)s_2(2)x_{12}(-3)\begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}x_{23}(1)x_{24}(-2) \\ &= s_2(0)s_1(1)s_2(2)x_{12}(-3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}x_{14}(5)x_{23}(1)x_{24}(-2). \end{aligned}$$

So

$$A = s_2(0)s_1(1)s_2(2)x_{12}(-3)\cdot 1_2 \cdot x_{23}(1)x_{14}(5)x_{24}(-2)$$

and $\mathrm{rank}(A) = 2$. □

IF14AQ0. Find the normal form of

$$A = \begin{pmatrix} -3 & -36 & -39 & -42 & -46 & -84 \\ -9 & -108 & -\frac{233}{2} & -114 & \frac{293}{2} & -167 \\ 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \end{pmatrix}.$$

This means, find $P \in GL_4(\mathbb{Q})$, $Q \in GL_6(\mathbb{Q})$ and $r \in \{0,1,2,3,4\}$ such that $A = P1_rQ$ and find R in reduced row echelon form such that $A = PR$.

Solution. *Step 1.* Let

$$s_1(c) = \begin{pmatrix} c & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_2(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_3(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

Then

$$\begin{aligned} A &= \begin{pmatrix} -3 & -36 & -39 & -42 & -46 & -84 \\ -9 & -108 & -\frac{233}{2} & -114 & \frac{293}{2} & -167 \\ 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \end{pmatrix} \\ &= s_2(-3) \begin{pmatrix} -3 & -36 & -39 & -42 & -46 & -84 \\ 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & \frac{1}{2} & 12 & \frac{23}{2} & -23 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \end{pmatrix} \\ &= s_2(-3)s_1(-1) \begin{pmatrix} 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & 0 & 0 & -1 & -36 \\ 0 & 0 & \frac{1}{2} & 12 & \frac{23}{2} & -23 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \end{pmatrix} \\ &= s_2(-3)s_1(-1)s_3(2) \begin{pmatrix} 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & 0 & 0 & -1 & -36 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \\ 0 & 0 & 0 & 0 & -1 & -36 \end{pmatrix} \\ &= s_2(-3)s_1(-1)s_3(2)s_2(0) \begin{pmatrix} 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \\ 0 & 0 & 0 & 0 & -1 & -36 \\ 0 & 0 & 0 & 0 & -1 & -36 \end{pmatrix} \\ &= s_2(-3)s_1(-1)s_3(2)s_2(0)s_3(1) \begin{pmatrix} 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \\ 0 & 0 & 0 & 0 & -1 & -36 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Step 2. Let

$$h_1(d) = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_2(d) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_3(d) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and let $x_{ij}(c)$ denote the square matrix with 1s on the diagonal, c in the (i,j) entry and 0 elsewhere.

$$\text{If}\qquad B=\begin{pmatrix}3&36&39&42&45&48\\0&0&\frac14&6&\frac{25}{4}&\frac{13}{2}\\0&0&0&0&-1&-36\\0&0&0&0&0&0\end{pmatrix}$$

then

$$\begin{aligned}B&=h_1(3)h_2(\tfrac14)h_3(-1)\begin{pmatrix}1&12&13&14&15&16\\0&0&1&24&25&26\\0&0&0&0&1&36\\0&0&0&0&0&0\end{pmatrix}\\&=h_1(3)h_2(\tfrac14)h_3(-1)\cdot x_{23}(25)x_{13}(15)x_{12}(13)\begin{pmatrix}1&12&0&14&0&16\\0&0&1&24&0&26\\0&0&0&0&1&36\\0&0&0&0&0&0\end{pmatrix}\\&=h_1(3)h_2(\tfrac14)h_3(-1)\cdot x_{23}(25)x_{13}(15)x_{12}(13)\cdot R\end{aligned}$$

where

$$\begin{aligned}R=\begin{pmatrix}1&12&0&14&0&16\\0&0&1&24&0&26\\0&0&0&0&1&36\\0&0&0&0&0&0\end{pmatrix}&=\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\end{pmatrix}\begin{pmatrix}1&12&0&14&0&16\\0&0&1&24&0&26\\0&0&0&0&1&36\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{pmatrix}\\&=1_3\begin{pmatrix}1&12&0&14&0&16\\0&0&1&24&0&26\\0&0&0&0&1&36\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{pmatrix}=1_3Q.\end{aligned}$$

So

$$\begin{aligned}B&=h_1(3)h_2(\tfrac14)h_3(-1)\cdot x_{23}(25)x_{13}(15)x_{12}(13)\cdot R\\&=h_1(3)h_2(\tfrac14)h_3(-1)\cdot x_{23}(25)x_{13}(15)x_{12}(13)\cdot 1_3\cdot Q,\end{aligned}$$

where

$$Q=\begin{pmatrix}1&12&0&14&0&16\\0&0&1&24&0&26\\0&0&0&0&1&36\\0&1&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0&12&14&16\\0&1&0&0&24&26\\0&0&1&0&0&36\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}s_3s_4s_2=\begin{array}{l}x_{36}(36)x_{26}(26)x_{16}(16)\\ \cdot x_{25}(24)x_{15}(14)x_{24}(12)\\ \cdot s_3s_4s_2.\end{array}$$

Thus,

$$\text{if}\qquad A=\begin{pmatrix}-3&-36&-39&-42&-46&-84\\-9&-108&-\frac{233}{2}&-114&\frac{293}{2}&-167\\3&36&39&42&45&48\\0&0&\frac14&6&\frac{25}{4}&\frac{13}{2}\end{pmatrix}\qquad\text{then}\qquad A=P1_3Q,$$

where
$$P = \begin{array}{c} s_2(-3)s_1(-1)s_3(2)s_2(0)s_3(1) \\ \cdot h_1(3)h_2(\frac{1}{4})h_3(-1)\cdot x_{23}(25)x_{13}(15)x_{12}(13) \end{array} \quad \text{in } GL_4(\mathbb{F})$$
and
$$Q = \begin{array}{c} x_{36}(36)x_{26}(26)x_{16}(16) \\ \cdot x_{25}(24)x_{15}(14)x_{24}(12) \\ \cdot s_3s_4s_2. \end{array} \quad \text{in } GL_6(\mathbb{F}).$$

□

For $c \in \mathbb{R}$, let
$$s_1(c) = \begin{pmatrix} c & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s_2(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s_3(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
For $c \in \mathbb{R}$, let
$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_{13}(c) = \begin{pmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
For c_1, c_2, c_3, c_4 nonzero elements of $\mathbb{R}$, let
$$d(c_1, c_2, c_3, c_4) = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{pmatrix}.$$
These notations will help us to write the row reduction process for
$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$
accurately and effficiently.

IF15AQ2. Find the normal form of
$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$
This means find $P \in GL_4(\mathbb{Q})$, $Q \in GL_5(\mathbb{Q})$ and $r \in \{0, 1, 2, 3, 4\}$ such that $A = P1_rQ$ and find R in reduced row echelon form such that $A = PR$.

Solution.

For $c \in \mathbb{R}$, let
$$s_1(c) = \begin{pmatrix} c & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s_2(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad s_3(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & c & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

For $c \in \mathbb{R}$, let

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_{13}(c) = \begin{pmatrix} 1 & 0 & c & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & c & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For c_1, c_2, c_3, c_4 nonzero elements of $\mathbb{R}$, let

$$d(c_1, c_2, c_3, c_4) = \begin{pmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \end{pmatrix}.$$

These notations will help us to write the row reduction process for

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

accurately and effficiently.

$$
\begin{aligned}
M &= \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & -1 & 1 & 1 \end{pmatrix} \\
&= s_3(1) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 3 & 1 & 1 \\ 0 & 3 & -1 & 1 & 1 \end{pmatrix} \\
&= s_3(1)s_2(1) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 3 & 1 & 1 \\ 0 & 3 & -1 & 1 & 1 \end{pmatrix} \\
&= s_3(1)s_2(1)s_1(0) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 3 & -1 & 1 & 1 \\ 0 & 0 & \frac{8}{3} & \frac{4}{3} & \frac{4}{3} \end{pmatrix} \\
&= s_3(1)s_2(1)s_1(0)s_3(-\tfrac{1}{3}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & -1 & 1 & 1 \\ 0 & 0 & \frac{4}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & \frac{8}{3} & \frac{4}{3} & \frac{4}{3} \end{pmatrix} \\
&= s_3(1)s_2(1)s_1(0)s_3(-\tfrac{1}{3})s_2(\tfrac{1}{3}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 3 & -1 & 1 & 1 \\ 0 & 0 & \frac{8}{3} & \frac{4}{3} & \frac{4}{3} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= s_3(1)s_2(1)s_1(0)s_3(-\tfrac{1}{3})s_2(\tfrac{1}{3})s_3(\tfrac{1}{2}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{3}{8} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= s_3(1)s_2(1)s_1(0)s_3(-\tfrac{1}{3})s_2(\tfrac{1}{3})s_3(\tfrac{1}{2})d(1, \tfrac{1}{3}, \tfrac{3}{8}, 1) \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \frac{4}{3} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= s_3(1)s_2(1)s_1(0)s_3(-\tfrac{1}{3})s_2(\tfrac{1}{3})s_3(\tfrac{1}{2})d(1, \tfrac{1}{3}, \tfrac{3}{8}, 1)x_{12}(1) \begin{pmatrix} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\
&= s_3(1)s_2(1)s_1(0)s_3(-\tfrac{1}{3})s_2(\tfrac{1}{3})s_3(\tfrac{1}{2})d(1, \tfrac{1}{3}, \tfrac{3}{8}, 1)x_{12}(1)x_{12}(\tfrac{4}{3}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$\square$

IF16HS53. Let $A \in M_{t\times s}(\mathbb{Q})$. Show that there exist $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$ and $r \in \{1, \ldots, \min(s,t)\}$ such that

$$A = P1_rQ.$$

where $1_r = E_{11} + E_{22} + \cdots + E_{rr}$ in $M_{t\times s}(\mathbb{Q})$.

Solution. FILL THIS OUT $\square$

IF17AQ6. Find the normal form of

$$A = \begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

This means find $P \in GL_4(\mathbb{Q})$, $Q \in GL_4(\mathbb{Q})$ and $r \in \{0,1,2,3,4\}$ such that $A = P1_rQ$ and find R in reduced row echelon form such that $A = PR$.

Solution.

Step 1:Using $s_i(c)$ to increasing the number of lower left 0 entries

$$\begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} = s_2(\tfrac{1}{8}) \begin{pmatrix} 7 & 6 & 2 & 4 \\ 8 & 6 & 3 & 5 \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$
$$= s_2(\tfrac{1}{8})s_1(\tfrac{7}{8}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & \frac{1}{4} & -\frac{3}{8} \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$
$$= s_2(\tfrac{1}{8})s_1(\tfrac{7}{8})s_3(\tfrac{29}{4}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & \frac{1}{4} & -\frac{3}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix}$$
$$= s_2(\tfrac{1}{8})s_1(\tfrac{7}{8})s_3(\tfrac{29}{4})s_2(\tfrac{3}{4}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{1}{2} & -\frac{15}{8} \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix}$$
$$= s_2(\tfrac{1}{8})s_1(\tfrac{7}{8})s_3(\tfrac{29}{4})s_2(\tfrac{3}{4})s_3(\tfrac{4}{5}) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \\ 0 & 0 & 0 & -\frac{71}{40} \end{pmatrix}$$

Step 2: Using $h_i(d)$ to make the first entry of each row equal to 1.

$$\begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \\ 0 & 0 & 0 & -\frac{71}{40} \end{pmatrix} = h_1(8)h_2(1)h_3(-\tfrac{5}{8})h_4(-\tfrac{71}{40}) \begin{pmatrix} 1 & \frac{6}{8} & \frac{3}{8} & \frac{5}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & \frac{49}{5} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Combining Step 1 and Step 2 gives

$$\begin{pmatrix}7&6&2&4\\1&8&7&9\\8&6&3&5\\0&1&1&2\end{pmatrix} = s_2(\tfrac{1}{8})s_1(\tfrac{7}{8})s_3(\tfrac{29}{4})s_2(\tfrac{3}{4})s_3(\tfrac{4}{5})h_1(8)h_2(1)h_3(-\tfrac{5}{8})h_4(-\tfrac{71}{40})\begin{pmatrix}1&\frac{6}{8}&\frac{3}{8}&\frac{5}{8}\\0&1&1&2\\0&0&1&\frac{49}{5}\\0&0&0&1\end{pmatrix}$$

Step 3: Using $x_{ij}(c)$ to make the first nonzero in each row equal to 0.

$$\begin{aligned}
\begin{pmatrix}1&\frac{6}{8}&\frac{3}{8}&\frac{5}{8}\\0&1&1&2\\0&0&1&\frac{49}{5}\\0&0&0&1\end{pmatrix} &= x_{34}(\tfrac{49}{5})\begin{pmatrix}1&\frac{6}{8}&\frac{3}{8}&\frac{5}{8}\\0&1&1&2\\0&0&1&0\\0&0&0&1\end{pmatrix} = x_{34}(\tfrac{49}{5})x_{24}(2)\begin{pmatrix}1&\frac{6}{8}&\frac{3}{8}&\frac{5}{8}\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\\
&= x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})\begin{pmatrix}1&\frac{6}{8}&\frac{3}{8}&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\\
&= x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})x_{23}(1)\begin{pmatrix}1&\frac{6}{8}&\frac{3}{8}&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\\
&= x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})x_{23}(1)x_{13}(\tfrac{3}{8})\begin{pmatrix}1&\frac{6}{8}&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\\
&= x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})x_{23}(1)x_{13}(\tfrac{3}{8})x_{12}(\tfrac{6}{8})\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}
\end{aligned}$$

Combining Step 1, Step 2 and Step 3 gives

$$\begin{pmatrix}7&6&2&4\\1&8&7&9\\8&6&3&5\\0&1&1&2\end{pmatrix} = \begin{array}{l}s_2(\tfrac{1}{8})s_1(\tfrac{7}{8})s_3(\tfrac{29}{4})s_2(\tfrac{3}{4})s_3(\tfrac{4}{5})\\ \quad\cdot h_1(8)h_2(1)h_3(-\tfrac{5}{8})h_4(-\tfrac{71}{40})\\ \qquad x_{34}(\tfrac{49}{5})x_{24}(2)x_{14}(\tfrac{5}{8})x_{23}(1)x_{13}(\tfrac{3}{8})x_{12}(\tfrac{6}{8}).\end{array}$$

□

IF18AQ7. (Row reducing a generic 2×1 matrix) Show that if $a, b \in \mathbb{Q}$ and $b \neq 0$ then $\begin{pmatrix}a\\b\end{pmatrix} = s_1(\frac{a}{b})\begin{pmatrix}b\\0\end{pmatrix}$.

Solution.

$$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}\frac{a}{b}&1\\1&0\end{pmatrix}\begin{pmatrix}b\\0\end{pmatrix} = s_1(\tfrac{a}{b})\begin{pmatrix}b\\0\end{pmatrix}.$$

□

IF19AQ8. (Row reducing a generic 3×3 matrix) Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be an element of $M_{3\times 3}(\mathbb{Q})$ and assume that $a_{31}\neq 0$ and $a_{22}-\frac{a_{32}a_{21}}{a_{31}}\neq 0$. Show that

$$A = s_2\Big(\frac{a_{21}}{a_{31}}\Big)s_1\Big(\frac{a_{11}}{a_{31}}\Big)s_2\Bigg(\frac{a_{12}-\frac{a_{32}a_{11}}{a_{31}}}{a_{22}-\frac{a_{32}a_{21}}{a_{31}}}\Bigg)\begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22}-\frac{a_{32}a_{21}}{a_{31}} & a_{23}-\frac{a_{33}a_{21}}{a_{31}} \\ 0 & 0 & a_{13}-\frac{a_{33}a_{11}}{a_{31}}-\frac{\left(a_{23}-\frac{a_{33}a_{21}}{a_{31}}\right)\left(a_{12}-\frac{a_{32}a_{11}}{a_{31}}\right)}{a_{22}-\frac{a_{32}a_{21}}{a_{31}}} \end{pmatrix}.$$

Solution.

$$\begin{aligned}
\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} &= s_2\Big(\frac{a_{21}}{a_{31}}\Big)\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ 0 & a_{22}-\frac{a_{32}a_{21}}{a_{31}} & a_{23}-\frac{a_{33}a_{21}}{a_{31}} \end{pmatrix} \\
&= s_2\Big(\frac{a_{21}}{a_{31}}\Big)s_1\Big(\frac{a_{11}}{a_{31}}\Big)\begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{12}-\frac{a_{32}a_{11}}{a_{31}} & a_{13}-\frac{a_{33}a_{11}}{a_{31}} \\ 0 & a_{22}-\frac{a_{32}a_{21}}{a_{31}} & a_{23}-\frac{a_{33}a_{21}}{a_{31}} \end{pmatrix} \\
&= s_2\Big(\frac{a_{21}}{a_{31}}\Big)s_1\Big(\frac{a_{11}}{a_{31}}\Big)s_2\Bigg(\frac{a_{12}-\frac{a_{32}a_{11}}{a_{31}}}{a_{22}-\frac{a_{32}a_{21}}{a_{31}}}\Bigg)\begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22}-\frac{a_{32}a_{21}}{a_{31}} & a_{23}-\frac{a_{33}a_{21}}{a_{31}} \\ 0 & 0 & a_{13}-\frac{a_{33}a_{11}}{a_{31}}-\frac{\left(a_{23}-\frac{a_{33}a_{21}}{a_{31}}\right)\left(a_{12}-\frac{a_{32}a_{11}}{a_{31}}\right)}{a_{22}-\frac{a_{32}a_{21}}{a_{31}}} \end{pmatrix}.
\end{aligned}$$

$\square$