7 Inner products

7.1 Standard inner products and lengths, distances and angles between vectors

IP1HS114. Let $u = |1+i, 1-i\rangle$ and $v = |i, 1\rangle$ in $\mathbb{C}^2$ with the standard inner product. Compute $\langle u|v\rangle$, $\langle v|u\rangle$, $\langle u|u\rangle$, $\langle u|u\rangle$, $d(u,v)$, $\cos(\theta(u,v))$ and $\theta(u,v)$.

Solution. Let $u = |1+i, 1-i\rangle$ and $v = |i, 1\rangle$ in $\mathbb{C}^2$ with the standard inner product. Then

$$\begin{aligned}
\langle u|v\rangle &= \langle 1+i, 1-i|i, 1\rangle = (1+i)\bar{i} + (1-i)\bar{1} \\
&= (1-i)(-i) + 1 - i = -i + 1 + 1 - i = 2 - 2i, \\
\langle v|u\rangle &= \langle i, 1|1+i, 1-i\rangle = i\overline{(1+i)} + 1\cdot\overline{(1-i)} \\
&= i(1-i) + 1 + i = i + 1 + 1 + i = 2 + 2i, \\
\langle u|u\rangle &= \langle 1+i, 1-i|1+i, 1-i\rangle = (1+i)\overline{(1+i)} + (1-i)\overline{(1-i)} \\
&= (1+i)(1-i) + (1-i)(1+i) = 1+1+1+1 = 4, \\
\langle u|u\rangle &= \langle i, 1 \mid i, 1\rangle = i\cdot\bar{i} + 1\cdot\bar{1} = i(-i) + 1 = 1 + 1 = 2, \\
d(u,v) &= \sqrt{\langle 1, -i|1, -i\rangle} = \sqrt{1\cdot\bar{1} + (-i)\overline{(-i)}} = \sqrt{1+1} = \sqrt{2}, \\
\cos(\theta(u,v)) &= \frac{\mathrm{Re}(\langle u|v\rangle)}{\|u\|\cdot\|v\|} = \frac{\mathrm{Re}(2-2i)}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}},
\end{aligned}$$

So $\theta(u,v) = \frac{\pi}{4}$. □

IP2HS115. Let $u = 1$ and $v = x$ in $\mathbb{R}[x]_{\le 2}$ with the standard inner product. Compute $\langle u|u\rangle$, $\langle v|v\rangle$, $\langle u|v\rangle$, $d(u,v)$, $\cos(\theta(u,v))$, and $\theta(u,v)$.

Solution. Let $u = 1$ and $v = x$ in $\mathbb{R}[x]_{\le 2}$ with the standard inner product. Then

$$\begin{aligned}
\langle u|u\rangle &= \langle 1|1\rangle = \int_0^1 dx = x\Big]_0^1 = 1, \\
\langle v|v\rangle &= \langle x|x\rangle = \int_0^1 x^2\,dx = \tfrac{1}{3}x^3\Big]_0^1 = \tfrac{1}{3} - 0 = \tfrac{1}{3}, \\
\langle u|v\rangle &= \langle 1|x\rangle = \int_0^1 x\,dx = \tfrac{1}{2}x^2\Big]_0^1 = \tfrac{1}{2} - 0 = \tfrac{1}{2}, \\
d(u,v) &= \sqrt{\langle x-1|x-1\rangle} = \sqrt{\int_0^1 (x-1)^2\,dx} \\
&= \sqrt{\tfrac{1}{3}(x-1)^3\Big]_0^1} = \sqrt{0 - \tfrac{1}{3}(-1)^3} = \sqrt{\tfrac{1}{3}} = \tfrac{1}{\sqrt{3}}, \\
\cos(\theta(u,v)) &= \frac{\mathrm{Re}(\langle u|v\rangle)}{\|u\|\cdot\|v\|} = \frac{\frac{1}{2}}{1\cdot\frac{1}{\sqrt{3}}} = \frac{\sqrt{3}}{2}.
\end{aligned}$$

So $\theta(u,v) = \frac{\pi}{6}$. □

IP3AQ12. Let $\langle,\rangle\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ be given by

$$\langle (u_1,u_2),(v_1,v_2)\rangle = u_1v_1 + 2u_2v_2 = \begin{pmatrix} u_1 & u_2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}.$$

Let $u = |3,1\rangle$ and $v = |-2,3\rangle$. Find $\|u\|$, $d(u,v)$ and $\theta(u,v)$.

Solution. If $z_1, z_2, \in \mathbb{R}$ then

$$\langle z_1, z_2 \| z_1, z_2\rangle = z_1^2 + 2z_2^2 \ge 0.$$

So $\langle,\rangle$ is positive definite since Let $u = |3,1\rangle$ and $v = |-2,3\rangle$. The length of u is

$$\|u\| = \sqrt{\langle u,u\rangle} = \sqrt{3\cdot 3 + 2\cdot 1\cdot 1} = \sqrt{9+2} = \sqrt{11}.$$

The distance between u and v is

$$d(u,v) = \|v-u\| = \sqrt{\langle(-5,2),(-5,2)\rangle} = \sqrt{25+2\cdot 4} = \sqrt{33}.$$

The angle between u and v is $\pi/2$ since

$$\cos(\theta(u,v)) = \frac{\langle u,v\rangle}{\|u\|\cdot\|v\|} = \frac{\langle(3,1),(-2,3)\rangle}{\sqrt{11}\cdot\sqrt{22}} = \frac{-6+2\cdot 3}{11\sqrt{2}} = 0 = \cos\left(\tfrac{\pi}{2}\right).$$

□

7.2 Gram matrices and positive definiteness

IP4HS116. Let $A \in M_{n\times n}(\mathbb{C})$ satisifying $A = \bar{A}^t$ and let $\langle,\rangle\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be given by

$$\langle (u_1,\ldots,u_n),(v_1,\ldots,v_n)\rangle = (u_1,\ldots,u_n)A\begin{pmatrix}\overline{v_1}\\ \vdots \\ \overline{v_n}\end{pmatrix} = u^tA\bar{v},$$

Prove that $\langle,\rangle$ satisfies all the properties of an inner product, except perhaps the positive definiteness.

Solution. If $u,v \in \mathbb{C}^n$ then

$$\begin{aligned}\overline{\langle v,u\rangle} &= \overline{(v^tA\bar{u})} = \overline{(\bar{u}^tA^tv)}^t = (u^t\bar{A}^t\bar{v})^t \\ &= (u^tA\bar{v})^t = \langle u,v\rangle,\end{aligned}$$

and if $\alpha \in \mathbb{C}$ and $u,v \in \mathbb{C}^n$ then

$$\langle \alpha u, v\rangle = (\alpha u)^tA\bar{v} = \alpha u^tAv = \alpha\langle u,v\rangle$$

and, if $u,v,w \in \mathbb{C}^n$ then

$$\begin{aligned}\langle u+v,w\rangle &= (u+v)^tA\bar{w} = (u^t+v^t)A\bar{w} \\ &= u^tA\bar{w} + v^tA\bar{w} = \langle u,w\rangle + \langle v,w\rangle.\end{aligned}$$

So $\langle,\rangle$ satisfies all the properties of an inner product, except perhaps the positive definiteness. □

IP5HS117. Is the map $\langle,\rangle\colon \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle (u_1,u_2,u_3),(v_1,v_2,v_3)\rangle = u_1v_1 - u_2v_2 + u_3v_3 \qquad \text{positive definite?}$$

Solution. The map $\langle,\rangle\colon \mathbb{R}^3\times\mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle (u_1,u_2,u_3),(v_1,v_2,v_3)\rangle = u_1v_1 - u_2v_2 + u_3v_3 = \begin{pmatrix} v_1 & v_2 & v_3\end{pmatrix}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$$

has
$$\langle (0,1,0),(0,1,0)\rangle = 0-1+0 = -1 \notin \mathbb{R}_{\geq 0}.$$
So $\langle,\rangle$ is not positive definite. $\square$

IP6HS118. Is the map $\langle,\rangle\colon \mathbb{C}^2\times\mathbb{C}^2 \to \mathbb{R}$ given by
$$\langle (u_1,u_2),(v_1,v_2)\rangle = iu_1\overline{v_1} - iu_2\overline{v_2} \qquad \text{positive definite?}$$

Solution. The map $\langle,\rangle\colon \mathbb{C}^2\times\mathbb{C}^2 \to \mathbb{R}$ given by
$$\langle (u_1,u_2),(v_1,v_2)\rangle = iu_1\overline{v_1} - iu_2\overline{v_2} = \begin{pmatrix}\overline{v_1} & \overline{v_2}\end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$
has
$$\langle (1,0),(1,0)\rangle = i\cdot 1\cdot \bar{1} - i\cdot 0\cdot \bar{0} = i \notin \mathbb{R}_{\geq 0}.$$
So $\langle,\rangle$ is not positive definite. $\square$

IP7HS119. Is the map $\langle,\rangle\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ given by
$$\langle (u_1,u_2),(v_1,v_2)\rangle = 2u_1v_1 - 2u_1v_2 - 2u_2v_1 + u_2v_2 \qquad \text{positive definite?}$$

Solution. The map $\langle,\rangle\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ given by
$$\langle (u_1,u_2),(v_1,v_2)\rangle = 2u_1v_1 - 2u_1v_2 - 2u_2v_1 + u_2v_2 = \begin{pmatrix} v_1 & v_2\end{pmatrix}\begin{pmatrix} 2 & -2 \\ -2 & 3\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$
has
$$\begin{aligned}\langle (u_1,u_2),(u_1,u_2)\rangle &= 2u_1^2 - 2u_2u_2 - 2u_2u_1 + 3u_2^2 \\ &= 2u_1^2 - 4u_1u_2 + 3u_2^2 \\ &= 2(u_1^2 - 2u_1u_2 + u_2^2) + u_2^2 \\ &= (u_1-u_2)^2 + u_2^2 \in \mathbb{R}_{\geq 0}.\end{aligned}$$
Assume $\langle (u_1,u_2),(u_1,u_2)\rangle = 0$.
Then $2(u_1-u_2)^2 + u_2^0 = 0$ then $u_2^2 = 0$ and $(u_1-u_2)^2 = 0$.
So $u_2 = 0$ and $u_1 = u_2 = 0$. So $(u_1,u_2) = 0$.

So $\langle,\rangle$ is positive definite. $\square$

IP8HS120. Is the map $\langle,\rangle\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ given by
$$\langle (u_1,u_2),(v_1,v_2)\rangle = u_1v_1 + 2u_2v_2 \qquad \text{positive definite?}$$

Solution. The map $\langle,\rangle\colon \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ given by
$$\langle (u_1,u_2),(v_1,v_2)\rangle = u_1v_1 + 2u_2v_2 = \begin{pmatrix} v_1 & v_2\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$
has $\langle (u_1,u_2),(u_1,u_2)\rangle = u_1^2 + 2u_2^2 \in \mathbb{R}_{\geq 0}$.
Assume $\langle (u_1,u_2),(u_1,u_2)\rangle = 0$.
Then $u_1^2 + 2u_2^0 = 0$ so that $u_1^2 = 0$ and $2u_2^2 = 0$.
So $(u_1,u_2) = 0$.

So $\langle , \rangle$ is positive definite. $\square$

IP9HS121. (Pythagorean theorem) Let V be an $\mathbb{F}$-vector space with an inner product. Let $u, v \in V$ and suppose that u and v are orthogonal. Prove that $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

Solution. Since u and v are orthogonal then

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle = \langle u, u+v\rangle + \langle v, u+v\rangle \\ &= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle \\ &= \|u\|^2 + 0 + 0 + \|v\|^2 \\ &= \|u\|^2 + \|v\|^2.\end{aligned}$$

$\square$

IP10HS122. Let V be a $\mathbb{F}$-vector space with an inner product $\langle , \rangle \colon V \times V \to \mathbb{F}$.
Let $B = \{b_1, \ldots, b_n\}$ be a basis of V and let A be the Gram matrix of $\langle , \rangle$ with respect to the basis B. Let $u, v \in V$. Prove that

$$\langle u, v\rangle = [u]_B^t A [\bar{v}]_B.$$

Solution. Let $u = u_1 b_1 + \cdots + u_n b_n \in V$ and let $v = v_1 b_1 + \cdots + v_n b_n \in V$ so that

$$[u]_B = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \quad \text{and} \quad [v]_B = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Then

$$\begin{aligned}\langle u, v\rangle &= \langle u_1 b_1 + \cdots + u_n b_n, v_1 b_1 + \cdots + v_n b_n\rangle \\ &= \sum_{i,j=1}^n u_i \overline{v_j} \langle b_i, b_j\rangle = \sum_{i,j=1}^n u_i A_{ij} \overline{v_j} \\ &= [u]_B^T A [\bar{v}]_B.\end{aligned}$$

$\square$

IP11HS123. Let $\langle \, , \, \rangle \colon \mathbb{R}[x]_{\le 2} \times \mathbb{R}[x]_{\le 2} \to \mathbb{R}$ be the standard inner product on the $\mathbb{R}$-vector space $\mathbb{R}[x]_{\le 2}$.

(a) Let $u = 7 + 3x + 2x^2$ and $v = 5 + x^2$. Calculate $\langle u|v\rangle$.

(b) Let A be the Gram matrix of $\langle \, , \, \rangle$ with respect to the basis $B = \{1, x, x^2\}$. Calculate A.

(c) Calculate $[u]_B$ and $[v]_B$ and $[u]_B^T A [v]_B$.

Solution. (a) Since the standard inner product $\langle \; | \; \rangle \colon \mathbb{R}[x]_{\le 2} \times \mathbb{R}[x]_{\le 2} \to \mathbb{R}$ has

$$\langle p|q\rangle = \int_0^1 pq \, dx$$

then

$$\begin{aligned}\langle u|v\rangle &= \int_0^1 (7 + 3x + 2x^2)(5 + x^2) \, dx \\ &= \int_0^1 (35 + 7x^2 + 15x + 3x^3 + 10x^2 + 2x^4) \, dx \\ &= 35 + \tfrac{7}{3} + \tfrac{15}{2} + \tfrac{3}{4} + \tfrac{10}{3} + \tfrac{2}{5}\end{aligned}$$

(b) Since

$$\begin{matrix} \langle 1|1\rangle = 1, & \langle 1|x\rangle = \frac{1}{2}, & \langle 1|x^2\rangle = \frac{1}{3}, \\ \langle x|1\rangle = \frac{1}{2}, & \langle x|x\rangle = \frac{1}{3}, & \langle x|x^2\rangle = \frac{1}{4}, \\ \langle x^2|1\rangle = \frac{1}{3}, & \langle x^2|x\rangle = \frac{1}{4}, & \langle x^2|x^2\rangle = \frac{1}{5}, \end{matrix}$$

then the Gram matrix of $\langle \ | \ \rangle$ with respect to B is

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

If $u = 7 + 3x + 2x^2$ and $v = 5 + x^2$ then

$$[u]_B = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad [v]_B = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}.$$

(c) By matrix multiplication,

$$\begin{aligned} [u]_B^t A [v]_B &= \begin{pmatrix} 7 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 3 & 2 \end{pmatrix} \begin{pmatrix} 5 + \frac{1}{3} \\ \frac{5}{2} + \frac{1}{4} \\ \frac{5}{3} + \frac{1}{5} \end{pmatrix} \\ &= 35 + \frac{7}{3} + \frac{15}{2} + \frac{3}{4} + \frac{10}{3} + \frac{2}{5}. \end{aligned}$$

So $\langle u|v\rangle = [u]_B^t A [v]_B$. □

IP12HS66. Let $u, v \in \mathbb{R}^n$ and let $Q \in O_n(\mathbb{R})$. Prove that $\langle Qu|Qv\rangle = \langle u|v\rangle$.

Solution. Let $u, v \in \mathbb{R}^n$ and let $Q \in O_n(\mathbb{R})$. Then

$$\begin{aligned} \langle u|v\rangle &= u^T v \quad \text{and} \\ \langle Qu|Qv\rangle &= (Qu)^T Qv = u^T Q^T Q v = u^T \cdot 1 \cdot v = u^T v. \end{aligned}$$

So $\langle Qu|Qv\rangle = \langle u|v\rangle$. □

7.3 Orthogonality and orthogonalization

IP13HS124. Let V be an $\mathbb{F}$-vector space with an inner product $\langle , \rangle \colon V \times V \to \mathbb{F}$. Assume $B = (b_1, \ldots, b_n)$ is an ordered orthonormal basis of V and $x \in V$. Prove that

$$x = \langle x, b_1\rangle b_1 + \cdots + \langle x, b_n\rangle b_n.$$

Solution. Let $x \in V$ and let $c_1, \ldots, c_n \in \mathbb{F}$ be such that $x = c_1 b_1 + \cdots + c_n b_n$.
If $j \in \{1, \ldots, n\}$ then

$$\begin{aligned} \langle x, b_j\rangle &= \langle c_1 b_1 + \cdots + c_n b_n, b_j\rangle = c_1\langle b_1, b_n\rangle + \cdots + c_n\langle b_n, b_j\rangle \\ &= c_1 \cdot 0 + \cdots + c_{j-1} \cdot 0 + c_j \cdot 1 + c_{j+1} \cdot 0 + \cdots + c_n \cdot 0 = c_j \cdot 1 = c_j. \end{aligned}$$

So $x = \langle x, b_1\rangle b_1 + \cdots + \langle x, b_n\rangle b_n$. □

IP14HS125. Let $\langle , \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be given by

$$\langle (u_1, u_2, u_3), (v_1, v_2, v_3)\rangle = u_1 v_1 + 2u_2 v_2 + u_3 v_3.$$

(a) Prove that $S = \{(1,1,1),(1-1,1),(1,0-1)\}$ is an orthogonal sequence in $\mathbb{R}^3$ with respect to $\langle , \rangle$.

(b) Let

$$\begin{aligned} b_1 &= \frac{1}{\|u\|}u, && \text{where} \quad u = (1,1,1), \\ b_2 &= \frac{1}{\|v\|}v, && \text{where} \quad v = (1,-1,1), \\ b_3 &= \frac{1}{\|w\|}w, && \text{where} \quad w = (1,0,01). \end{aligned}$$

Show that $\{b_1, b_2, b_3\}$ is an orthonormal sequence in $\mathbb{R}^3$ with respect to $\langle , \rangle$.

(c) Let $x = |1,1,-1\rangle$ and $c_2, c_2, c_3 \in \mathbb{R}$ such that

$$x = c_1b_1 + c_2b_2 + c_3b_3.$$

Compute c_1, c_2, c_3.

Solution. (a) Since

$$\begin{aligned} \langle (1,1,1),(1,-1,1)\rangle &= 1-2+1 = 0, \\ \langle (1,1,1),(1,0,-1)\rangle &= 1+0-1 = 0, \\ \langle (1,-1,1),(1,0,-1)\rangle &= 1+0-1 = 0, \end{aligned}$$

then S is an orthogonal sequence in $\mathbb{R}^3$ with respect to $\langle , \rangle$.

(b) Let

$$\begin{aligned} b_1 &= \frac{1}{\|u\|}u, && \text{where} \quad u = (1,1,1), \\ b_2 &= \frac{1}{\|v\|}v, && \text{where} \quad v = (1,-1,1), \\ b_3 &= \frac{1}{\|w\|}w, && \text{where} \quad w = (1,0,01). \end{aligned}$$

Then

$$\begin{aligned} b_1 &= \tfrac{1}{2}(1,1,1) && \text{since} & \langle (1,1,1),(1,1,1)\rangle &= 4, \\ b_2 &= \tfrac{1}{2}(1,-1,1) && \text{since} & \langle (1,-1,1),(1,-1,1)\rangle &= 4, \\ b_2 &= \tfrac{1}{\sqrt{2}}(1,0,-1) && \text{since} & \langle (1,0,-1),(1,0,-1)\rangle &= 2, \end{aligned}$$

and $\{b_1, b_2, b_3\}$ is an orthonormal sequence in $\mathbb{R}^3$ with respect to $\langle , \rangle$.

(c) Let $x = |1,1,-1\rangle$ and $c_2, c_2, c_3 \in \mathbb{R}$ such that

$$x = c_1b_1 + c_2b_2 + c_3b_3.$$

Then

$$\begin{aligned} c_1 &= c_1\langle b_1, b_1\rangle + 0 + 0 = \langle c_1b_1 + c_2b_2 + c_3b_3, b_1\rangle = \langle x, b_1\rangle \\ &= \langle (1,1,-1), \tfrac{1}{2}(1,1,1)\rangle = \tfrac{1}{2}(1+2-1), \\ c_2 &= c_2\langle b_2, b_2\rangle = \langle c_1b_1 + c_2b_2 + c_3b_3, b_2\rangle = \langle x, b_2\rangle \\ &= \langle (1,1,-1), \tfrac{1}{2}(1,-1,1)\rangle = \tfrac{1}{2}(1-2-1), \\ c_3 &= \langle x, b_3\rangle = \langle (1,1,-1), \tfrac{1}{\sqrt{2}}(1,0,-1)\rangle = \tfrac{1}{\sqrt{2}}(1+0+1) = \tfrac{2}{\sqrt{2}} = \sqrt{2}. \end{aligned}$$

So x is written as a linear combination of the basis elements in the form

$$\begin{aligned} x = (1,1,-1) &= \langle x, b_1\rangle b_1 + \langle x, b_2\rangle b_2 + \langle x, b_3\rangle b_3 \\ &= 1\cdot b_1 + (-1)\cdot b_2 + \sqrt{2} b_3 \\ &= (1,1,1) - (1,-1,1) + \sqrt{2}(1,0,-1). \end{aligned}$$

□

IP15HS126. Let $V = \mathbb{R}^n$ and let $u \in V$ with $u \neq 0$. Let

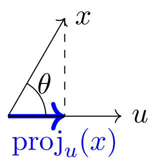
$$W = \mathbb{R}\text{-span}\{u\} = \{au \mid a \in \mathbb{R}\}.$$

(a) Show that W is a subspace of V.

(b) Find an orthonormal basis of W.

(c) Let $x \in V$. Prove that

$$\mathrm{proj}_W(x) = \mathrm{proj}_u(x).$$



Solution. (a) (aa) Since $0 \cdot u = 0$ then $0 \in W$.
(ab) Assume $w_1, w_2 \in W$. Then there exist $a_1, a_2 \in \mathbb{R}$ such that $w_1 = a_1 u$ and $w_2 = a_2 u$. Then

$$w_1 + w_2 = a_1 u + a_2 u = (a_1 + a_2)u. \qquad \text{So } w_1 + w_2 \in W,$$

(ac) Assume $w \in W$ and $c \in \mathbb{R}$. Then there exists $a \in \mathbb{R}$ such that $w = au$. Then

$$cw = c(aw) = (ca)w. \qquad \text{So } cw \in W.$$

So W is a subspace of $\mathbb{R}^n$.
(b) Let

$$b_1 = \frac{1}{\|u\|} u.$$

Then $\{b_1\}$ is an orthonormal basis of W.
(c) Let $x \in V$. Then

$$\mathrm{proj}_W(x) = \langle x, b_1\rangle b_1 = \langle x, \frac{1}{\|u\|} u\rangle \frac{1}{\|u\|} u = \frac{\langle x, u\rangle}{\|u\|^2} u = \mathrm{proj}_u(x).$$

□

IP16HS127. Let $V = \mathbb{R}^3$ and let $W = \{|x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}$.

(a) Show that the set

$$\{b_1, b_2\} = \left\{\tfrac{1}{\sqrt{2}}|1,-1,0\rangle, \tfrac{1}{\sqrt{6}}|1,1,-2\rangle\right\}$$

is an orthonormal basis of W with respect to the standard inner product on $\mathbb{R}^3$.

(b) Let $x = |1,2,3\rangle$. Compute $\mathrm{proj}_W(x)$.

(c) Compute the shortest distance from x to W.

Solution. (a) Since

$$\begin{aligned}
\langle b_1, b_1\rangle &= \langle \tfrac{1}{\sqrt{2}}(1,-1,0), \tfrac{1}{\sqrt{2}}(1,-1,0)\rangle = \tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{2}}\langle 1,-1,0\ |1,-1,0\rangle \\
&= \tfrac{1}{2}(1\cdot 1 + (-1)\cdot(-1) + 0\cdot 0) = \tfrac{1}{2}\cdot 2 = 1, \\
\langle b_1, b_2\rangle &= \langle \tfrac{1}{\sqrt{2}}(1,-1,0), \tfrac{1}{\sqrt{6}}(1,1,2)\rangle = \tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{6}}\langle 1,-1,0\ |1,1,2\rangle \\
&= \tfrac{1}{2\sqrt{3}}(1\cdot 1 + (-1)\cdot 1 + 0\cdot 2) = \tfrac{1}{2\sqrt{3}}\cdot 0 = 0, \\
\langle b_2, b_2\rangle &= \langle \tfrac{1}{\sqrt{6}}(1,1,2), \tfrac{1}{\sqrt{6}}(1,1,2)\rangle = \tfrac{1}{\sqrt{6}}\cdot\tfrac{1}{\sqrt{6}}\langle 1,1,2\ |1,1,2\rangle \\
&= \tfrac{1}{6}(1\cdot 1 + 1\cdot 1 + 2\cdot 2) = \tfrac{1}{6}\cdot 6 = 1,
\end{aligned}$$

then $\{b_1, b_2\}$ is orthonormal.

Since $1 + -1 + 0 = 0$ and $1 + 1 + (-2) = 0$ then b_1 and b_2 are elements of V. If $c_1, c_2 \in W$ and $c_1b_1 + c_2b_2 = 0$ then

$$\begin{aligned}
0 &= \tfrac{1}{\sqrt{2}}c_1 + \tfrac{1}{\sqrt{6}}c_2, \\
0 &= -\tfrac{1}{\sqrt{2}}c_1 + \tfrac{1}{\sqrt{6}}c_2, \qquad \text{which gives } c_1 = 0 \text{ and } c_2 = 0. \\
0 &= 0\cdot c_1 - \tfrac{2}{\sqrt{6}}c_2,
\end{aligned}$$

So $\{b_1, b_2\}$ are linearly independent. So $\{b_1, b_2\}$ is a basis of W.

(b) Let $x = |1,2,3\rangle$. Then

$$\begin{aligned}
\mathrm{proj}_W(x) &= \langle x|b_1\rangle b_1 + \langle x|b_2\rangle b_2 \\
&= \langle 1,2,3|\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\rangle\cdot\tfrac{1}{\sqrt{2}}|1,-1,0\rangle \\
&\quad + \langle 1,2,3|\tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}}\rangle\cdot\tfrac{1}{\sqrt{6}}|1,1,-2\rangle \\
&= \big(\tfrac{1}{\sqrt{2}} - \tfrac{2}{\sqrt{2}}\big)\tfrac{1}{\sqrt{2}}|1,-1,0\rangle + \tfrac{1}{6}(1+2-6)|1,1,-2)\rangle \\
&= |\tfrac{-1}{2}, \tfrac{1}{2}, 0\rangle + |\tfrac{-1}{2}, \tfrac{-1}{2}, 1) = |-1,0,1\rangle.
\end{aligned}$$

(c) The shortest distance from x to W is

$$\begin{aligned}
\|x - \mathrm{proj}_W(x)\| &= \|\, |1,2,3\rangle - |-1,0,1\rangle\, \| \\
&= \|\, |2,2,2\rangle\, \| = \sqrt{4+4+4} = 2\sqrt{3}.
\end{aligned}$$

□

IP17HS128. Let $V = \mathbb{R}^3$ with the standard inner product. Let $S = \{v_1, v_2, v_3\}$ with

$$v_1 = |1,1,1\rangle, \quad v_2 = |0,1,1\rangle, \quad v_1 = |0,0,1\rangle.$$

Apply the Gram-Schmidt process to convert S into an orthonormal basis B.

Solution. *Step 1.* Make v_1 into a unit vector. Let

$$b_1 = \frac{1}{\|v_1\|}v_1 = \tfrac{1}{\sqrt{3}}|1,1,1\rangle = |\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\rangle$$

and let $S = \{b_1, v_2, v_3\}$.
Step 2. Make v_2 orthogonal to b_1. Let

$$\begin{aligned} u_2 &= v_2 - \langle v, b_1 \rangle b_1 \\ &= |0,1,1\rangle - \tfrac{2}{\sqrt{3}} |\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\rangle \\ &= |\tfrac{-2}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\rangle \end{aligned}$$

and let $S_2 = \{b_1, u_2, v_3\}$.
Step 3. Make u_2 into a unit vector. Let

$$b_2 = \frac{1}{\|u_2\|} u_2 = \tfrac{1}{\sqrt{6/3}} |\tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\rangle = |\tfrac{-2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\rangle$$

and let $S_3 = \{b_1, b_2, v_3\}$.
Step 4. Make v_3 orthogonal to b_1 and b_2. Let

$$\begin{aligned} u_3 &= v_3 - \langle v_3, b_1 \rangle b_1 - \langle v_3, b_2 \rangle b_2 \\ &= |0,0,1\rangle - \tfrac{1}{\sqrt{3}} |\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\rangle - \tfrac{1}{\sqrt{6}} |\tfrac{-2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\rangle \\ &= |\tfrac{-1}{3} + \tfrac{2}{6}, \tfrac{-1}{3} - \tfrac{1}{6}, 1 - \tfrac{1}{3} - \tfrac{1}{6}\rangle = |0, \tfrac{-1}{2}, \tfrac{1}{2}\rangle. \end{aligned}$$

Step 5. Make u_3 into a unit vector. Let

$$b_3 = \frac{1}{\|u_3\|} u_3 = \tfrac{1}{\sqrt{2/4}} |0, \tfrac{-1}{2}, \tfrac{1}{2}\rangle = |0, \tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle.$$

Then

$$B = \{b_1, b_2, b_3\} \quad \text{is an orthonormal set.}$$

$\square$

IP18HS129. Let V be a $\mathbb{C}$-vector space with inner product $\langle , \rangle : V \times V \to \mathbb{C}$.
Let $B = \{b_1, \ldots, b_k\}$ be an orthogonal set in V. Prove that B is linearly independent.

Solution. Assume B is an orthogonal set in V.
To show: B is linearly independent.
To show: If $c_1, \ldots, c_k \in \mathbb{C}$ and $c_1 b_1 + \cdots + c_k b_k = 0$

$$\text{then } c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

Assume $c_1, \ldots, c_k \in \mathbb{C}$ and $c_1 b_1 + \cdots + c_k b_k = 0$.
To show: $c_1 = 0, c_2 = 0, \ldots, c_k = 0$.
To show: If $i \in \{1, \ldots, k\}$ then $c_i = 0$.
Assume $i \in \{1, \ldots, k\}$. To show: $c_i = 0$.

$$\begin{aligned} 0 &= \langle c_1 b_1 + \cdots + c_k b_k, b_i \rangle \\ &= c_1 \langle b_1, b_i \rangle + \cdots + c_{i-1} \langle b_{i-1}, b_i \rangle + c_i \langle b_i, b_i \rangle \\ &\qquad + c_{i+1} \langle b_{i+1}, b_i \rangle + \cdots + c_k \langle b_k, b_i \rangle \\ &= c_1 \cdot 0 + \cdots + c_{i-1} \cdot 0 + c_i \langle b_i, b_i \rangle \\ &\qquad + c_{i+1} \cdot 0 + \cdots + c_k \cdot 0 \\ &= c_i \langle b_i, b_i \rangle. \end{aligned}$$

Since $\langle , \rangle$ is an inner product and $b_i \neq 0$ then $\langle b_i, b_i\rangle \neq 0$. So

$$c_i = \frac{1}{\langle b_i, b_i\rangle} \cdot 0 = 0.$$

So B is linearly independent. □

IP19AQ11. (Gram-Schmidt with polynomials) Assume that $\mathbb{R}[x]_{\le 2}$ has the inner product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$

Starting with the basis $\{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$ find an orthonormal basis of $\mathbb{R}[x]_{\le 2}$.

Solution. Let $v_1 = 1$. Since

$$\|1\|^2 = \langle 1, 1\rangle = \int_0^1 dx = 1 \qquad \text{then} \qquad u_1 = \frac{1}{1}1 = 1.$$

Then

$$w_2 = x - \langle x, 1\rangle u_1 = x - \frac{1}{2} \cdot 1 = x - \frac{1}{2}$$

and

$$\|w_2\|^2 = \langle x - \frac{1}{2}, x - \frac{1}{2}\rangle = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{3}(x - \frac{1}{2})^3\Big]_0^1 = \frac{1}{3}(\frac{1}{8} + \frac{1}{8}) = \frac{1}{12}.$$

So

$$u_2 = 2\sqrt{3}(x - \frac{1}{2}).$$

FINISH THIS. □

IP20AQ14. (Gram-Schmidt) Use the standard inner product on $\mathbb{R}^3$ and apply Gram-Schmidt process to the set

$$S = \{|1, 1, 1\rangle, |0, 1, 1\rangle, |0, 0, 1\rangle\}.$$

Analyze the process to show that

$$P^tAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{where} \quad P = h_1(\tfrac{1}{\sqrt{3}})x_{12}(\tfrac{-2}{\sqrt{3}})h_2(\tfrac{3}{\sqrt{2}})x_{23}(\tfrac{-1}{\sqrt{6}})x_{13}(\tfrac{-1}{\sqrt{3}})h_3(\tfrac{1}{\sqrt{2}}).$$

Solution. Since $\langle 1, 1, 1|1, 1, 1\rangle = 3$ then

$$u_1 = \tfrac{1}{\sqrt{3}}\,|1, 1, 1\rangle.$$

Letting $v_2 = |0, 1, 1\rangle$ then $\langle v_2|u_1\rangle = \frac{2}{\sqrt{3}}$ and

$$w_2 = |0, 1, 1\rangle - \tfrac{2}{\sqrt{3}}\tfrac{1}{\sqrt{3}}\,|1, 1, 1\rangle = \tfrac{1}{3}\,|-2, 1, 1\rangle,$$

and

$$u_2 = \tfrac{1}{\sqrt{6}}\,|-2, 1, 1\rangle.$$

Then $\langle v_3, u_1\rangle = \frac{1}{\sqrt{3}}$ and $\langle v_3|u_2\rangle = \frac{1}{\sqrt{6}}$ and

$$w_3 = |0, 0, 1\rangle - \tfrac{1}{\sqrt{3}}\tfrac{1}{\sqrt{3}}|1, 1, 1\rangle - \tfrac{1}{\sqrt{6}}\tfrac{1}{\sqrt{6}}|-2, 1, 1\rangle = |0, -\tfrac{1}{2}, \tfrac{1}{2}\rangle.$$

Then

$$u_3 = \frac{1}{\sqrt{2}}\,|0,-1,1\rangle.$$

The inital Gram matrix is

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The first change of basis matrix is

$$[I]_{S_1S} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = h_1\big(\tfrac{1}{\sqrt{3}}\big) \qquad \text{and} \qquad A_2 = \begin{pmatrix} 1 & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{3}} & 2 & 1 \\ \frac{1}{\sqrt{3}} & 1 & 1 \end{pmatrix} = [I]^t_{SS_1}A[I]_{SS_1}.$$

The second change of basis matrix is

$$[I]_{S_1S_2} = \begin{pmatrix} 1 & -\frac{2}{\sqrt{3}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_{12}\big(\tfrac{-2}{\sqrt{3}}\big) \qquad \text{and} \qquad A_3 = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{6}{9} & \frac{1}{3} \\ \frac{1}{\sqrt{3}} & \frac{1}{3} & 1 \end{pmatrix} = [I]^t_{S_1S_2}A_2[I]_{S_1S_2}.$$

The third change of basis matrix is

$$[I]_{S_1S_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = h_2\big(\tfrac{3}{\sqrt{2}}\big) \qquad \text{and} \qquad A_4 = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 1 & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & 1 \end{pmatrix} = [I]^t_{S_2S_3}A_3[I]_{S_2S_3}.$$

The fourth change of basis matrix is

$$[I]_{S_3S_4} = \begin{pmatrix} 1 & 0 & -\frac{1}{\sqrt{3}} \\ 0 & 1 & -\frac{1}{\sqrt{6}} \\ 0 & 0 & 1 \end{pmatrix} = x_{23}\big(\tfrac{-1}{\sqrt{6}}\big)x_{13}\big(\tfrac{-1}{\sqrt{3}}\big) \qquad \text{and} \qquad A_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = [I]^t_{S_3S_4}A_4[I]_{S_3S_4}.$$

The last change of basis matrix is

$$[I]_{S_4S_5} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = h_3\big(\tfrac{1}{\sqrt{2}}\big) \qquad \text{and} \qquad A_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [I]^t_{S_4S_5}A_5[I]_{S_4S_5}.$$

$\square$