8 Geometry in $\mathbb{R}^n$

GE1HS1. Let

$$e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Compute $3e_1 + 5e_2 + (-2)e_3 + 0e_4$.

Solution.

$$3e_1 + 5e_2 + (-2)e_3 + 0e_4. = 3\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + 5\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + (-2)\begin{pmatrix}0\\0\\1\\0\end{pmatrix} + 0\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\5\\-2\\0\end{pmatrix}$$

$\square$

GE2HS2. Let $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that

$$\langle y, x\rangle = \langle x, y\rangle, \qquad \langle x, y\rangle = x^T y, \qquad \|x\| = \sqrt{\langle x, x\rangle}, \qquad d(x, y) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}.$$

Solution. If

$$x = |x_1, x_2, \ldots, x_n\rangle \quad \text{and} \quad y = |y_1, y_2, \ldots, y_n\rangle$$

then

$$\langle x, y\rangle = x^T y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

and

$$\|x\| = \sqrt{\langle x, x\rangle} \quad \text{and} \quad \|x\|^2 = \langle x, x\rangle \qquad \text{and}$$

$$d(x, y) = \|y - x\| = \sqrt{\langle y - x, y - x\rangle} = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}.$$

$\square$

GE3HS3. Let $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that

$$\langle y, x\rangle = \langle x, y\rangle, \qquad \langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle, \qquad \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$
$$\langle x, cy\rangle = c\langle x, y\rangle, \qquad \langle cx, y\rangle = c\langle x, y\rangle, \qquad \|cx\| = |c| \cdot \|x\|.$$

Solution. Let $x, y, z \in \mathbb{R}^n$ and let $c \in \mathbb{R}$.

$$\begin{aligned}\langle y, x\rangle &= y_1x_1 + y_2x_2 + \cdots + y_nx_n \\ &= x_1y_1 + x_2y_2 + \cdots x_ny_n = \langle x, y\rangle,\end{aligned}$$

$$\langle x, y + z\rangle = x^T(y + z) = x^T y + x^T z = \langle x, y\rangle + \langle x, z\rangle,$$

$$\langle x + y, z\rangle = \langle z, x + y\rangle = \langle z, x\rangle + \langle z, y\rangle = \langle x, z\rangle + \langle y, z\rangle,$$

$$\langle x, cy\rangle = x^T cy = cx^T y = c\langle x, y\rangle, \langle cx, y\rangle = \langle y, cx\rangle = c\langle y, x\rangle = c\langle x, y\rangle,$$

$$\|cx\| = \sqrt{\langle cx, cx\rangle} = \sqrt{c^2\langle x, x\rangle} = \sqrt{c^2}\sqrt{\langle x, x\rangle} = |c|\cdot\|x\|.$$

□

GE4HS4. Let $x, y \in \mathbb{R}^n$. Prove that

$$|\langle x, y\rangle| \le \|x\|\cdot\|y\|.$$

Solution. Let $x, y \in \mathbb{R}^n$. If $x = 0$ then both sides of the Cauchy-Schwarz inequality are 0. Assume $x \neq 0$. The Gram-Schmidt process on the vectors (x, y) suggests the consideration of

$$u_1 = \frac{x}{\|x\|} \qquad \text{and} \quad u_2 = y - \frac{\langle y, x\rangle}{\langle x, x\rangle}x.$$

To avoid denominators, let $u = \langle x, x\rangle y - \langle y, x\rangle x$. Then

$$\begin{aligned} 0 \le \langle u, u\rangle &= \big\langle \langle x, x\rangle y - \langle y, x\rangle x, \langle x, x\rangle y - \langle y, x\rangle x\big\rangle \\ &= \overline{\langle x, x\rangle}\langle x, x\rangle|\langle y, y\rangle - \langle x, x\rangle\overline{\langle y, x\rangle}\langle y, x\rangle - \langle y, x\rangle\overline{\langle x, x\rangle}\langle x, y\rangle + |\langle y, x\rangle|^2\langle x, x\rangle \\ &= \overline{\langle x, x\rangle}(\langle x, x\rangle\langle y, y\rangle - |\langle y, x\rangle|^2) \end{aligned}$$

Since $x \neq 0$ then $\langle x, x\rangle \in \mathbb{K}_{>0}$ and so $\overline{\langle x, x\rangle} = \langle x, x\rangle \in \mathbb{K}_{>0}$. Thus,

$$0 \le \langle x, x\rangle\langle y, y\rangle - |\langle y, x\rangle|^2 \qquad \text{and so} \qquad |\langle y, x\rangle|^2 \le \langle x, x\rangle\langle y, y\rangle.$$

Using the theorem that if $z, w \in \mathbb{R}_{\ge 0}$ then $z^2 \le w^2$ if and only if $z \le w$,

$$|\langle x, y\rangle| \le \|x\|\cdot\|y\|.$$

□

GE5HS5. Let $x, y \in \mathbb{R}^n$. Prove that

$$\|x + y\| \le \|x\| + \|y\|$$

Solution. Let $a \in \mathbb{F}$. Using that if $z \in \mathbb{F}$ then $|z|^2 = z\bar{z} \in \mathbb{R}_{\ge 0}$, then

$$|a + \bar{a}|^2 \le |a + \bar{a}|^2 + |a - \bar{a}|^2 = (a + \bar{a})^2 - (a - \bar{a})^2 = 4a\bar{a} = 4|a|^2.$$

So $|a + \bar{a}| \le 2|a|$. Also

$$\text{if } a + \bar{a} \in \mathbb{R}_{\le 0} \text{ then } a + \bar{a} \le 0 \le |a + \bar{a}| \qquad \text{and} \qquad \text{if } a + \bar{a} \in \mathbb{K}_{\ge 0} \text{ then } a + \bar{a} = |a + \bar{a}|.$$

Combining these with $|a + \bar{a}| \le 2|a|$ gives

$$a + \bar{a} \le 2|a|.$$

Assume $x, y \in V$. Then

$$\begin{aligned} \|x + y\|^2 &= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\ &= \|x\|^2 + |y\|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} \\ &\le \|x\|^2 + |y\|^2 + 2|\langle x, y\rangle| \\ &\le \|x\|^2 + |y\|^2 + 2\|x|\cdot\|y\| \\ &= (\|x\| + \|y\|)^2. \end{aligned}$$

Thus, $\|x+y\| \le \|x\| + \|y\|$. $\square$

GE6HS6. Let $u = |1,3,1,2\rangle$ and $v = |2,1,-1,3\rangle$ in $\mathbb{R}^4$. Compute $u-v$ and $d(u,v)$.
Solution. If $u = |1,3,1,2\rangle$ and $v = |2,1,-1,3\rangle$ in $\mathbb{R}^4$ then

$$u - v = |1,-2,0,1\rangle$$

and the distance between the points $(1,3,1,2)$ and $(2,1,-1,3)$ is

$$d(u,v) = \|\ |1,-2,0,1\rangle\ \| = \sqrt{1^2+(-2)^2+0^2+1^2} = \sqrt{1+4+0+1} = \sqrt{6}.$$

$\square$

GE7HS7. Let $u = |0,2,2,-1\rangle$ and $v = |-1,1,1,-1\rangle$ in $\mathbb{R}^4$. Compute $\langle u,v\rangle$, $\|u\|$ and $\|v\|$ and check that $|\langle u,v\rangle| \le \|u\| \cdot \|v\|$.
Solution. If $u = |0,2,2,-1\rangle$ and $v = |-1,1,1,-1\rangle$ in $\mathbb{R}^4$ then

$$\begin{aligned}\langle u,v\rangle &= \langle 0,2,2,-1|-1,1,1,-1\rangle \\ &= 0\cdot(-1) + 2\cdot 1 + 2\cdot 1 + (-1)\cdot(-1) \\ &= 0+2+1+1 = 5\end{aligned}$$

and

$$\|u\| = \sqrt{0+4+4+1} = \sqrt{9} = 3 \qquad \text{and} \qquad \|v\| = \sqrt{1+1+1+1} = \sqrt{4} = 2.$$

Since $|5| \le 3\cdot 2$ we observe that, in this case,

$$|\langle u,v\rangle| \le \|u\| \cdot \|v\|.$$

$\square$

GE8HS8. Let $u = (2,-1,-2)$ and $v = (2,1,3)$. Find vectors v_1 and v_2 such that

$$v = v_1 + v_2$$

where v_1 is parallel to u and v_2 is perpendicular to u.

Solution: Since the projection of v onto u is parallel to u then let

$$\begin{aligned}v_1 = \mathrm{proj}_u v = \frac{\langle u,v\rangle}{\langle u,u\rangle} u &= \frac{-3}{9} u \\ &= \tfrac{-1}{3}|2,-1,-2\rangle = |\tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\rangle\end{aligned}$$

and

$$v_2 = u - v_1 = |2,-1,-2\rangle - |\tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\rangle = |\tfrac{8}{3}, \tfrac{2}{3}, \tfrac{7}{3}\rangle.$$

Then $u = v_1 + v_2$ and v_1 is parallel to u and v_2 is perpendicular to u. $\square$

GE9HS9. Determine the vector, parametric and Cartesian equations of the line through the points $P = (-1,2,3)$ and $Q = (4,-2,5)$.

Solution: Since the direction of the line is

$$Q - P = |4,-2,5\rangle - |-1,2,3\rangle = |5,-4,2\rangle$$

and

$$P = |-1, 2, 3\rangle \text{ is a point on the line}$$

then the line is the set of points in $\mathbb{R}^3$ given by

$$\{\, |-1, 2, 3\rangle + t \cdot |5, -4, 2\rangle \mid t \in \mathbb{R}\}.$$

Parametric equations for the line are

$$\begin{array}{l} x = -1 + 5t, \\ y = 2 - 4t, \\ z = 3 + 2t, \end{array} \qquad \text{with } t \in \mathbb{R}.$$

Solving for t, the Cartesian equation of the line is

$$\frac{x+1}{5} = \frac{y-2}{-4} = \frac{z-3}{2}.$$

$\square$

GE10HS10. Find a vector equation of the 'friendly' line through the point $(2, 0, 1)$ that is parallel to the 'enemy' line

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-6}{2}.$$

Does the point $(0, 4, -3)$ lie on the 'friendly' line?

Solution: Letting

$$t = \frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-6}{2}$$

gives

$$\begin{array}{l} x = 1 + t, \\ y = -2 - 2t, \\ z = 6 + 2t \end{array} \quad \text{with } t \in \mathbb{R},$$

and

$$\{|1, -2, 6\rangle + t|1, -2, 2\rangle \mid t \in \mathbb{R}\}$$

is the set of points in $\mathbb{R}^3$ that lie on the 'enemy' line.

The 'friendly' line we want is parallel to the 'enemy' line and goes through the point $|2, 0, 1\rangle$.

So the 'friendly' line consists of the points

$$\{\, |2, 0, 1\rangle + t\, |1, -2, 2\rangle \mid t \in \mathbb{R}\}.$$

Since

$$|2, 0, 1\rangle + (-2) \cdot |1, -2, 2\rangle = |0, 4, -3\rangle$$

then $|0, 4, -3\rangle$ is on the 'friendly' line. $\square$

GE11HS11. Find the vector equation for the plane in $\mathbb{R}^3$ containing the points $P = |1, 0, 2\rangle$ and $Q = |1, 2, 3\rangle$ and $R = |4, 5, 6\rangle$.

Solution: The point $|1, 0, 2\rangle$ is in the plane and two vectors in the plane are

$$Q - P = |0, 2, 1\rangle \quad \text{and} \quad R - P = |3, 5, 4\rangle.$$

So the points in the plane are the points $|x, y, z\rangle$ in $\mathbb{R}^3$ which satisfy

$$|x, y, z\rangle = |1, 0, 2\rangle + s|0, 2, 1\rangle + t|3, 5, 4\rangle \quad \text{with } s, t \in \mathbb{R}.$$

GE12HS12. Where does the line
$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$
intersect the plane $3x + 2y + z = 20$?

Solution: The line in parametric form is

$$\begin{matrix} x = 1 + t, \\ y = 2 + 2t, \\ z = 3 + 3t, \end{matrix} \qquad \text{with } t \in \mathbb{R},$$

and plugging into the equation of the plane gives

$$20 = 3(t + 1) + 2(2t + 2) + (3t + 3) = 10t + 10 \text{ so that } t = 1.$$

Thus the point $|x, y, z\rangle$ with $x = 1 + 1 = 2$, $y = 2 + 2 = 4$ and $z = 3 + 3$ is on both the line and the plane.

GE13HS13. Find a vector form for the line of intersection of the two planes $x + 3y + 2z = 6$ and $3x + 2y + z = 11$.

Solution: The points on the intersection of the two planes are the points $|x, y, z\rangle$ that satisfy the system of equations
$$\begin{matrix} 3x + 2y - z = 11, \\ x + 3y + 2z = 6. \end{matrix}$$

One of the main points of this course is to learn how to use matrices as an efficient and organized mechanics for solving systems of equations of this type. For now, let's proceed ad hoc. The second equation gives
$$x = 6 - 3y - 2z, \quad \text{and plugging back into} \quad 3x + 2y - z = 11$$
gives
$$11 = 3(6 - 3y - 2z) + 2y - z = 18 - 9y - 6z + 2y - z = 18 - 7y - 7z.$$
So $7y = 7 - 7z$ and $y = 1 - z$. So $x = 6 - 3y - z = 6 - 3(1 - z) - 2z$ and

$$\begin{matrix} x = 3 + z, \\ y = 1 - z, \\ z = 0 + z, \end{matrix} \qquad \text{where } z \text{ can be any number.}$$

So the line is the set of points $|x, y, z\rangle$ such that

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad \text{with } z \in \mathbb{R}.$$

So the line is
$$p + \mathbb{R}v, \quad \text{where } p = |3, 1, 0\rangle \text{ and } v = |1, -1, 1\rangle.$$

□

GE14HS14. Let $\hat{\imath}, \hat{\jmath}, \hat{k} \in \mathbb{R}^3$ be given by

$$\hat{\imath} = |1,0,0\rangle, \qquad \hat{\jmath} = |0,1,0\rangle, \qquad \hat{k} = |0,0,1\rangle.$$

Compute $5\hat{\imath} + (-1)\hat{\jmath} + (-4)\hat{k}$.

Solution: $5\hat{\imath} + (-1)\hat{\jmath} + (-4)\hat{k} = |5,-1,-4\rangle$. □

GE15HS15. Let $u = |u_1,u_2,u_3\rangle, v = |v_1,v_2,v_3\rangle, w = |w_1,w_2,w_3\rangle$ in $\mathbb{R}^3$. Prove that

$$u \times v = \det\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}\hat{\imath} - \det\begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}\hat{\jmath} + \det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\hat{k}, \qquad \text{and}$$

$$\langle u, v \times w\rangle = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Solution: Using the definition of $u \times v$ and the definition of the determinant,

$$\begin{aligned} u \times v &= (u_2v_3 - u_3v_2)\hat{\imath} + (u_3v_1 - u_1v_3)\hat{\jmath} + (u_1v_2 - u_2v_1)\hat{k} \\ &= \det\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix}\hat{\imath} - \det\begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix}\hat{\jmath} + \det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\hat{k} \end{aligned}$$

and

$$\begin{aligned} \langle u, v \times w\rangle &= \langle u_1, u_2, u_3 | (v_2w_3 - w_3v_2, -(v_1w_3 - v_3w_1), v_1w_2 - v_2w_1\rangle \\ &= u_1(v_2w_3 - v_3w_2) - u_2(v_1w_3 - v_3w_1) + u_3(v_1w_2 - v_2w_1) \\ &= \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}. \end{aligned}$$

□

GE16HS16. Let $u = |u_1,u_2,u_3\rangle, v = |v_1,v_2,v_3\rangle, w = |w_1,w_2,w_3\rangle$ In $\mathbb{R}^3$. Prove that

$$\langle v, v \times w\rangle = 0 \qquad \text{and} \qquad \langle w, v \times w\rangle = 0.$$

Conclude that $v \times w$ is perpendicular to both v and w.

Solution: Since

$$\langle v, v \times w\rangle = \det\begin{pmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = 0 \quad \text{and} \quad \langle w, v \times w\rangle = \det\begin{pmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = 0$$

then

$$v \times w \text{ is perpendicular to both } v \text{ and } w.$$

□

GE17H17. Find a vector perpendicular to both $|1,1,1\rangle$ and $|1,-1,-2\rangle$.

Solution. By definition of the cross product

$$\begin{aligned} &|1,1,1\rangle \times |1,-1,-2\rangle \\ &\quad = |1 \cdot (-2) - 1 \cdot (-1), -(1 \cdot (-2) - 1 \cdot 1), 1 \cdot (-1) - 1 \cdot 1\rangle \\ &\quad = |-1, 3, -2\rangle. \end{aligned}$$

The vector $|-1,3,-2\rangle$ is perpendicular to both $|1,1,1\rangle$ and $|1,-1,-2\rangle$ since

$$\langle -1,3,-2 \mid 1,1,1\rangle = -1+3-2 = 0$$

and

$$\langle -1,3,-2 \mid 1,-1,-2\rangle = -1-3+4 = 0.$$

$\square$

GE18HS18. Find the volume of the parallelipiped with adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$, where

$$P = |2,0,-1\rangle, \qquad Q = |4,1,0\rangle, \qquad R = |3,-1,1\rangle \qquad \text{and} \qquad S = |2,-2,2\rangle.$$

Solution. Since the edges of the parallelipiped are

$$\overrightarrow{PQ} = P - Q = |2,1,1\rangle, \qquad \overrightarrow{PR} = P - R = |1,-1,2\rangle, \qquad \overrightarrow{PS} = P - S = |0,-2,3\rangle,$$

then

$$\begin{aligned}
(\text{Volume of parallelipiped}) &= |\langle \overrightarrow{PQ}, \overrightarrow{PR} \times \overrightarrow{PS}\rangle| \\
&= \left| \det \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{pmatrix} \right| = \left| 2 \cdot \det \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} - \det \begin{pmatrix} 1 & 1 \\ -2 & 3 \end{pmatrix} \right| \\
&= |2(-3+4) - (3+2)| = |-3| = 3.
\end{aligned}$$

$\square$

GE19HS19. Find the area of the triangle in $\mathbb{R}^3$ with vertices $|2,-5,4\rangle$, $|3,-4,5\rangle$ and $|3,-6,2\rangle$.

Solution. Letting

$$u = |3,-4,5\rangle - |2,-5,4\rangle = |1,1,1\rangle \qquad \text{and} \qquad v = |3,-6,2\rangle - |2,-5,4\rangle = |1,-1,-2\rangle,$$

then

$$\begin{aligned}
u \times v &= |1,1,1\rangle \times |1,-1,-2\rangle \\
&= |1\cdot(-2) - 1\cdot(-1), -(1\cdot(-2) - 1\cdot 1), 1\cdot(-1) - 1\cdot 1\rangle \\
&= |-1,3,-2\rangle.
\end{aligned}$$

Then

$$\begin{aligned}
(\text{Area of triangle}) &= \tfrac{1}{2}(\text{area of rectangle with edges } u \text{ and } v) \\
&= \tfrac{1}{2} \frac{1}{\|u \times v\|} \begin{pmatrix} \text{volume of parallelipiped} \\ \text{with edges } u,\ v \text{ and } u \times v \end{pmatrix} \\
&= \tfrac{1}{2} \frac{1}{\|u \times v\|} \langle u \times v, u \times v\rangle \\
&= \tfrac{1}{2} \|u \times v\| = \tfrac{1}{2} \|\, |-1,3,-2\rangle\, \| \\
&= \tfrac{1}{2} \sqrt{(-1)^2 + 3^2 + (-2)^2} = \frac{\sqrt{14}}{2}.
\end{aligned}$$

$\square$

GE20HS20. Find the Cartesian equation of the plane with vector form

$$|x,y,z\rangle = s\,|1,-1,0\rangle + t\,|2,0,1\rangle + |-1,1,1\rangle, \text{ with } s,t\in\mathbb{R}.$$

Solution. A normal vector to this plane is

$$n = u\times v, \qquad \text{where } u = |1,-1,0\rangle \text{ and } v = |2,0,1\rangle.$$

Then $n = u\times v = |-1-0, -(1-0), 0-(-2)\rangle = |-1,-1,2\rangle$.
Then $|-1,1,1\rangle$ is a point in the plane, and

$$\langle -1,1,1 \mid u\times v\rangle = \langle -1,1,1 \mid -1,-1,2\rangle = 1-1+2 = 2.$$

Since the plane is

$$|-1,1,1\rangle + \{|x,y,z\rangle \in \mathbb{R}^3 \mid \langle x,y,z \mid -1,-1,2\rangle = 0\}$$

then the Cartesian equation of the plane is

$$-x-y+2z = 2.$$

$\square$