4 Eigenvectors, eigenvalues and diagonalization

4.1 Eigenvalues and eigenvectors

EV1HS56. Let $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find $P, D \in GL_2(\mathbb{Q})$ such that D is diagonal and $A = PDP^{-1}$.

Solution. First,

$$\begin{aligned} A - t &= \begin{pmatrix} 1-t & 4 \\ 1 & 1-t \end{pmatrix} = \begin{pmatrix} 1-t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1-t \\ 0 & 4-(1-t)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1-t & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1-t \\ 0 & -(t+1)(t-3) \end{pmatrix} \end{aligned}$$

Case 1: $t + 1 = 0$. Then

$$A + 1 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \ker(A+1) = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$$

Case 2: $t - 3 = 0$. Then

$$A - 3 = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \ker(A-3) = \text{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

$$\text{If} \quad P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \text{ then } P^{-1} = \frac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$\begin{aligned} \text{and} \quad PDP^{-1} &= \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \cdot \tfrac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \tfrac{1}{4} \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 6 \end{pmatrix} \\ &= \tfrac{1}{4} \begin{pmatrix} 4 & 16 \\ 4 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} = A. \end{aligned}$$

The characteristic polynomial of A is

$$\det(A - t) = \det(D - t) = (-1-t)(3-t).$$

EV2HS57. Let $A = \begin{pmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find $P, D \in GL_2(\mathbb{Q})$ such that D is diagonal and $A = PDP^{-1}$.

Solution. Find $\ker(A-t)$ by row reduction:

$$\begin{aligned}
A-t &= \begin{pmatrix} 2-t & -3 & 6 \\ 0 & 5-t & -6 \\ 0 & 1 & 0-t \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5-t & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2-t & -3 & 6 \\ 0 & 1 & -t \\ 0 & 0 & -6-(5-t)(-t) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5-t & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2-t & 0 & 3(2-t) \\ 0 & 1 & -t \\ 0 & 0 & -(t^2-5t+6) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3 & 0 \\ 0 & 5-t & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2-t & 0 & 3(2-t) \\ 0 & 1 & -t \\ 0 & 0 & -(t-2)(t-3) \end{pmatrix}.
\end{aligned}$$

Case 1: $t-2=0$. Then

$$A-2 = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 5-2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\ker(A-2) = \ker \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right\}.$$

Case 2: $t-3=0$. Then

$$\begin{aligned}
A-3 &= \begin{pmatrix} 1 & 3 & 0 \\ 0 & 5-3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2-3 & 0 & 3(2-3) \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

and

$$\ker(A-3) = \ker \begin{pmatrix} -1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{span}\left\{ \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Then $A = PDP^{-1}$ where

$$P = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$\text{and} \quad P^{-1} = -\begin{pmatrix} -1 & 0 & 0 \\ -3 & 1 & -1 \\ 6 & -3 & 2 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 & -6 \\ 0 & -1 & 3 \\ 0 & 1 & -2 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\det(A-t) = \det(D-t) = (2-t)^2(3-t).$$

□

EV3HS58. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ viewed as an element of $M_{2\times 2}(\mathbb{R})$. Show that A does not have any eigenvalues or eigenvectors. Conclude that A is not diagonalizable as an element of $M_{2\times 2}(\mathbb{R})$.

Solution. The linear transformation

$$\begin{matrix} T\colon & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & v & \mapsto & Av \end{matrix} \quad \text{is a rotation of } \tfrac{3\pi}{2} \text{ about } (0,0).$$

There are no nonzero vectors that point in the same direction after a rotation of by an angle of $\frac{3\pi}{2}$. So, as an element of $M_{2\times 2}(\mathbb{R})$, the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{has no eigenvalues and no eigenvectors.}$$

So, as an element of $M_{2\times 2}(\mathbb{R})$, the matrix A is not diagonalizable. □

EV4HS59. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ viewed as an element of $M_{2\times 2}(\mathbb{C})$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find $P, D \in GL_2(\mathbb{C})$ such that D is diagonal and $A = PDP^{-1}$.

Conclude that A is diagonalizable as an element of $M_{2\times 2}(\mathbb{C})$.

Solution. As an element of $M_{2\times 2}(\mathbb{C})$, the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{has two eigenvalues, } i \text{ and } -i.$$

$$\ker(A - i) = \mathrm{span}\left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\} \qquad \text{and} \qquad \ker(A + i) = \mathrm{span}\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}.$$

$\begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue i and

$\begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigenvector of eigenvalue $-i$.

If

$$P = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ then } P^{-1} = \frac{1}{-2i}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$$

and

$$\begin{aligned} PDP^{-1} &= \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \cdot \frac{1}{-2i}\begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix} \\ &= \tfrac{1}{2}i \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} i & 1 \\ i & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = A. \end{aligned}$$

The characteristic polynomial of A is

$$\det(A - t) = \det(D - t) = (i - t)(-i - t) = t^2 + 1.$$

□

EV5HS60. Find a matrix $A \in M_{2\times 2}(\mathbb{C})$ that does not have 2 linearly independent eigenvectors. Conclude that A is not diagonalizable.

Solution.

$$\text{Let} \qquad A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}. \qquad \text{Then} \qquad A - t = \begin{pmatrix} 1-t & 2 \\ 0 & 1-t \end{pmatrix}$$

which has a single row of 0s when $t = 1$. (The characteristic polynomial of A is $\det(A-t) = (1-t)^2$.)

$$\ker(A-1) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.$$

Since A does not have two linearly independent eigenvectors then

$$A \text{ is not diagonalizable.}$$

□

4.2 Diagonalization

EV6HS79. Let $D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Compute D^{10}.

Solution.

$$\text{If} \qquad D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{then} \qquad D^{10} = \begin{pmatrix} (-4)^{10} & 0 & 0 \\ 0 & 3^{10} & 0 \\ 0 & 0 & 2^{10} \end{pmatrix}.$$

□

EV7HS80. Assume $A = PDP^{-1}$. Prove that if $k \in \mathbb{Z}_{>0}$ then $A^k = PD^kP^{-1}$.

Solution. If $A = PDP^{-1}$ then

$$\begin{aligned} A^3 &= A \cdot A \cdot A = (PDP^{-1})(PDP^{-1})(PDP^{-1}) = PDP^{-1}PDP^{-1}PDP^{-1} \\ &= PD \cdot D \cdot DP^{-1} = PD^3P^{-1}, \end{aligned}$$

This is the idea, let us write a proper proof by induction.
Base case: k=1. This is $A = PDP^{-1}$ (which is assumed in the statement of the question)..
Induction step. Assume that $k \in \mathbb{Z}_{>0}$ and $A = PDP^{-1}$ and $A^k = PD^kP^{-1}$.
To show: $A^{k+1} = PD^{k+1}P^{-1}$.

$$A^{k+1} = A^k \cdot A = (PD^kP^{-1}) \cdot (PDP^{-1} = PD^kP^{-1}PDP^{-1} = PD^kDP^{-1} = PD^{k+1}P^{-1}.$$

Thus, if $A = PDP^{-1}$ and $k \in \mathbb{Z}_{>0}$ then $A^k = PD^kP^{-1}$. □

EV8HS80b. Assume $A = PDP^{-1}$ and D is invertible. Prove that if $k \in \mathbb{Z}_{>0}$ then $A^{-k} = PD^{-k}P^{-1}$.

Solution. Since both D and P are invertible then $A = PDP^{-1}$ 9s invertible with

$$A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}.$$

Then, if $k \in \mathbb{Z}_{>0}$ then

$$A^{-k} = (A^{-1})^k = (PD^{-1}P^{-1})^k = P(D^{-1})^kP^{-1} = PD^{-k}P^{-1}.$$

□

EV9HS81. Let

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}.$$

(a) Compute P^{-1} and prove that $A = PDP^{-1}$.

(b) Compute A^k for $k \in \mathbb{Z}$.

Solution. Let

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}.$$

Then

$$P^{-1} = \tfrac{1}{4}\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad A = PDP^{-1}.$$

So

$$\begin{aligned} A^k = PD^kP^{-1} &= \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^k & 0 \\ 0 & 3^k \end{pmatrix} \tfrac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} (-1)^k \cdot 2 & 3^k \cdot 2 \\ (-1)^k & 3^k \end{pmatrix} \tfrac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \tfrac{1}{4} \begin{pmatrix} 2((-1)^k + 3^k) & 4((-1)^{k+1} + 3^k) \\ (-1)^{k+1} + 3^k & 2((-1)^k + 3^k) \end{pmatrix} \end{aligned}$$

□

EV10HS82. Let

$$x_n = \begin{pmatrix} r_n \\ p_n \\ w_n \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

(a) Show that

$$\text{if} \quad P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{then} \quad P^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}.$$

(b) Show that $T = PDP^{-1}$.

(c) Let $r_0, p_0, w_0 \in \mathbb{R}$ and let

$$x_0 = \begin{pmatrix} r_0 \\ p_0 \\ w_0 \end{pmatrix}. \qquad \text{Compute } \lim_{n\to\infty} T^n x_0.$$

(d) Explain how this is used in applications to genetics.

Solution. Let

$$x_n = \begin{pmatrix} r_n \\ p_n \\ w_n \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

and define an evolution process by

$$x_{n+1} = Tx_n.$$

This is the *Markov chain* defined by T. Since $T = PDP^{-1}$, where

$$P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \ D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ P^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

then *the stationary state of the process on* $\mathbb{R}^3$ *defined by* T is

$$\begin{aligned} \lim_{n\to\infty} T^n x_0 &= \lim_{n\to\infty} PD^nP^{-1}x_0 = \lim_{n\to\infty} P \begin{pmatrix} 1^n & 0 & 0 \\ 0 & (\frac{1}{2})^n & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}x_0 \\ &= P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}x_0 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_0 \\ &= \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} x_0 = \begin{pmatrix} \frac{1}{4}(r_0 + p_0 + w_0) \\ \frac{1}{2}(r_0 + p_0 + w_0) \\ \frac{1}{4}(r_0 + p_0 + w_0) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}. \end{aligned}$$

(d) An example of the type of problem in genetics which this dynamical system models is the following setup.

A snapdragon plant has either red, pink or white flowers, depending on its genotype (with alleles RR, RW or WW respectively).

We want to cross pink plants with red, pink and white plants. On average,

- crossing pink plants with red plants produces:

$$\tfrac{1}{2} \text{ red plants and } \tfrac{1}{2} \text{ pink plants as offspring.}$$

- crossing pink plants with pink plants produces:

$$\tfrac{1}{4} \text{ red plants, } \tfrac{1}{2} \text{ pink plants and } \tfrac{1}{4} \text{ white plants as offspring.}$$

- crossing pink plants with white plants produces:

$$\tfrac{1}{2} \text{ pink plants and } \tfrac{1}{2} \text{ white plants as offspring.}$$

This is an example of incomplete dominance in genetics.

If you continue crossing with only pink plants, what fractions of the three types of flowers would you eventually expect to see in your garden?

The matrix T used in this question contains the proportions of plants that are obtained ineach successive generation and the process of multiplying by T^k models what happens after breeding for k generations. The limit $\lim_{n\to infty} T^n x_0$models what eventually is produced by x_0 after many generations. $\square$

4.3 Diagonalization with symmetric, hermitian, orthogonal and unitary matrices

EV11HS67. Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Find an orthogonal matrix $Q \in O_2(\mathbb{R})$ and a diagonal matrix $D \in M_{2\times 2}(\mathbb{R})$ such that $A = QDQ^T$.

Solution. The characteristic polynomial of the symmetric matrix

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{is} \quad \begin{aligned} \det(A-t) &= (1-t)^2 - 1 \\ &= 1 - 2t + t^2 - 1 = t^2 - 2t \\ &= (t-0)(t-2). \end{aligned}$$

Then

$$\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

are eigenvectors of length 1 with eigenvalues 0 and 2, respectively.

Then

$$Q = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{is orthogonal} \qquad \text{and} \qquad A = Q\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}Q^T.$$

□

EV12HS68. Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Find an unitary matrix $Q \in U_2(\mathbb{C})$ and a diagonal matrix $D \in M_{2\times 2}(\mathbb{C})$ such that $A = QDQ^T$.

Solution. The characteristic polynomial of the Hermitian matrix

$$A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \quad \text{is} \quad \begin{aligned} \det(A-t) &= (1-t)^2 - (-i)\cdot i \\ &= 1 - 2t + t^2 - 1 = t^2 - 2t \\ &= (t-0)(t-2). \end{aligned}$$

Then

$$\tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

are eigenvectors of A of length 1 with eigenvalues 0 and 2, respectively. Then

$$U = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \text{is unitary} \qquad \text{and} \qquad A = U\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\bar{U}^T.$$

□

EV13HS69. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Find $U \in O_2(\mathbb{R})$, $V \in O_3(\mathbb{R})$ and $S \in M_{2\times 3}(\mathbb{R})$ such that S is diagonal and $A = USV^T$.

Solution. Since

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{then} \quad A^TA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The columns v_1, v_2, v_3 of

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{are orthonormal eigenvectors of } A^TA$$

with eigenvalues $\lambda = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$. Let

$$u_1 = \frac{1}{\sqrt{2}} A v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_2 = \frac{1}{\sqrt{1}} A v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then $\{u_1, u_2\}$ is already an orthonormal basis of $\mathbb{R}^2$, Let

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{1} & 0 \end{pmatrix}$$

Then $U \in O_2(\mathbb{R})$, $V \in O_3(\mathbb{R})$ and $S \in M_{2\times 3}(\mathbb{R})$ and $A = USV^T$. $\square$

EV14HS70. Let $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. Find $U \in O_2(\mathbb{R})$, $V \in O_2(\mathbb{R})$ and $S \in M_{2\times 2}(\mathbb{R})$ such that S is diagonal and $A = USV^T$.

Solution. Since

$$A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{then} \quad A^T A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The columns v_1, v_2 of

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{are orthonormal eigenvectors of } A^T A$$

with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 0$. Let

$$u_1 = \frac{1}{\sqrt{1}} A v_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and let} \quad u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

so that $\{u_1, u_2\}$ is an orthonormal basis of $\mathbb{R}^2$, Let

$$U = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} \sqrt{1} & 0 \\ 0 & \sqrt{0} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Then $U \in O_2(\mathbb{R})$, $V \in O_2(\mathbb{R})$ and $S \in M_{2\times 2}(\mathbb{R})$ and $A = USV^T$. $\square$