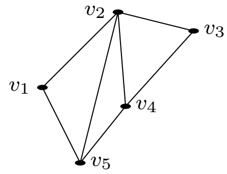
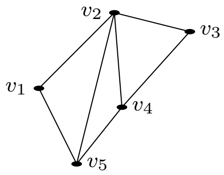
9 Additional applications

AP1HS77. Use adjacency matrices to compute the number of paths of length 3 from vertex i to vertex j in the following graph for every pair (i, j) with $i, j \in \{1, \ldots, 5\}$.



Solution. The graph



has adjacency matrix $A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$

There is a 1 in the (i, j) entry if there is an edge connecting vertex i and vertex j.
Then

$$A^3 = A(A^2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & 2 & 1 \\ 1 & 4 & 1 & 2 & 2 \\ 1 & 1 & 2 & 1 & 2 \\ 2 & 2 & 1 & 3 & 1 \\ 1 & 2 & 2 & 1 & 3 \end{pmatrix}$$

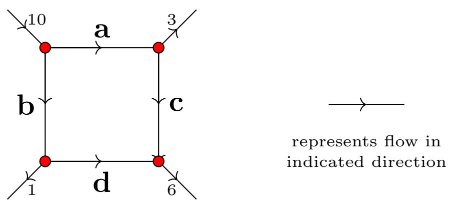
$$= \begin{pmatrix} 2 & 6 & 3 & 3 & 5 \\ 6 & 6 & 6 & 7 & 7 \\ 3 & 6 & 2 & 5 & 3 \\ 3 & 7 & 5 & 4 & 7 \\ 5 & 7 & 3 & 7 & 4 \end{pmatrix}.$$

The (i, j) entry of A^3 gives the number of paths of length three from vertex i to vertex j. □

AP2HS78. At each node • require

$$(\text{sum of flows in}) = (\text{sum of flows out}).$$

Determine a, b, c and d in the network



Solution. Requiring (sum of flows in) = (sum of flows out) at each node • gives

$$\begin{array}{ll} \text{Node 1:} & 10 = a+b, \\ \text{Node 2:} & a = 3+c, \\ \text{Node 3:} & c+d = 6, \\ \text{Node 4:} & b = 1+d. \end{array}$$

So

$$\begin{array}{r} a+b+0c+0d=10, \\ a+0b-c+0d=2, \\ 0a+0b+c+d=6, \\ 0a+b+0c-d=1 \end{array} \quad \text{which is} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix}.$$

Start with

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 6 \\ 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 1 \\ 6 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \\ 6 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \\ 0 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix}.$$

This is the system

$$\begin{matrix} a = 9, \\ b - d = 1, \\ c + d = 6, \\ 0 = 0, \end{matrix} \qquad \text{which is} \qquad \begin{matrix} a = & 9 & + & 0d, \\ b = & 1 & + & 1d, \\ c = & 6 & + & (-1)d, \\ d = & 0 & + & 1d. \end{matrix}$$

where d can be any number. So

$$\begin{aligned} \mathrm{Sol}(Ax = b) &= \left\{ \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; d \in \mathbb{Q} \right\} \\ &= \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + \mathbb{Q}\text{-span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

AP3HS138. Suppose that the data set of assignment and exam marks for 7 students is

Student	*AssignmentMark*	*ExamMark*
*S*1	99	100
*S*2	80	82.5
*S*3	79	79
*S*4	75.5	82.5
*S*5	87.5	91
*S*6	67	67.5
*S*7	76	68

Calculate the mean assignment mark μ_A, the mean exam mark μ_E and the and the correlation $\mathrm{corr}(A, E) = \cos(\theta(A - \mu_A, E - \mu_E))$ between the assignment marks and the exam marks.

Solution. The mean assignment mark is

$$\mu_A = \tfrac{1}{7}(99 + 80 + 79 + 75.5 + 87.5 + 67 + 76) = 80.5.$$

The mean exam mark is

$$\mu_E = \tfrac{1}{7}(100 + 82.5 + 79 + 82.5 + 91 + 67.5 + 68) = 81.5.$$

Then

$$\begin{aligned} A - \mu_A &= |18.5, -0.5, -1.5, -5.5, 7, -13.5, -4.5\rangle, \\ E - \mu_E &= |18.5, 1, -2.5, 1, 9.5, -14, -13.5\rangle \end{aligned}$$

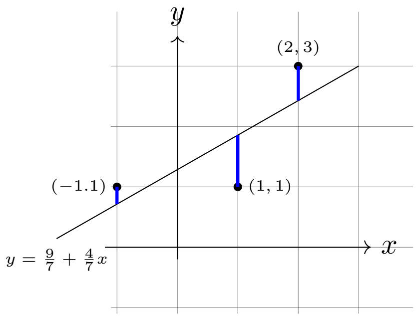
and the correlation between the assignment marks and the exam marks is

$$\begin{aligned} \mathrm{corr}(A, E) &= \cos(\theta(A - \mu_A, E - \mu_E)) \\ &= \frac{\langle A - \mu_A, E - \mu_E\rangle}{\|A - \mu_A\| \cdot \|E - \mu_E\|} = \frac{656.75}{(24.92)(28.62)} \approx 0.92. \end{aligned}$$

$\square$

AP4HS139. Given the data set $D = \{(-1,1),(1,1),(2,3)\}$ let

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$



Compute

$$\vec{s} = (A^tA)^{-1}A^t\vec{y} = \begin{pmatrix} a_{\text{best}} \\ b_{\text{best}} \end{pmatrix}$$

and determine the list of best fit $y = a_{\text{best}} + b_{\text{best}}x$ for the data set D.

Solution.

$$A^tA = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix} \quad \text{and}$$

$$A^t\vec{y} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} \quad \text{and} \quad (A^tA)^{-1} = \tfrac{1}{14}\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}.$$

So

$$\vec{s} = (A^tA)^{-1}A^t\vec{y} = \frac{1}{14}\begin{pmatrix} 6 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ 6 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 18 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{9}{7} \\ \frac{4}{7} \end{pmatrix}.$$

So the line of best fit is

$$y = \tfrac{9}{7} + \tfrac{4}{7}x.$$

$\square$