13 Kernels, images and solving linear equations – Assignment questions

13.1 Solving systems of linear equations

KE1HS162. Find the values of $a, b \in \mathbb{Q}$ for which the system

$$\begin{aligned} x - 2y + z &= 4, \\ 2x - 3y + z &= 7, \\ 3x - 6y + az &= b, \end{aligned} \qquad \text{has} \qquad \begin{array}{l} \text{(a) no solution,} \\ \text{(b) a unique solution,} \\ \text{(c) LOTS of solutions.} \end{array}$$

KE2HS39. (The number of solutions of a linear system)

(a) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

(b) Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

(c) Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

KE3HS40. Let $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $x = \begin{pmatrix} x \\ y \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Compute $\mathrm{Sol}(Ax = b)$.

KE4HS41. Solve the following system of linear equations.

$$\begin{aligned} 4x - 2y + 5z &= 31, \\ 2x - 3y - 2z &= 13, \\ x - 3y + 2z &= 11. \end{aligned}$$

KE5HS42. Let $A \in GL_n(\mathbb{Q})$ and let $x, b \in \mathbb{Q}^n$. Compute $\mathrm{Sol}(Ax = b)$.

KE6HS43. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$.

(a) Compute $\mathrm{Sol}(Ax = b)$.

(b) Compute $\mathrm{Sol}(Ax = 0)$.

(c) Compute $\ker(A)$.

(d) Compute $\mathrm{im}(A)$.

KE7HS44. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ 15 \end{pmatrix}$.

(a) Compute $\mathrm{Sol}(Ax = b)$.

(b) Compute $\mathrm{Sol}(Ax = 0)$.

(c) Compute $\ker(A)$.

(d) Compute $\mathrm{im}(A)$.

KE8HS45. Let $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(a) Compute $\mathrm{Sol}(Ax = b)$.

(b) Compute $\mathrm{Sol}(Ax = 0)$.

(c) Compute $\ker(A)$.

(d) Compute $\mathrm{im}(A)$.

KE9HS46. Let $S = \{|1, 3, -1, 1\rangle,\ |2, 6, 0, 4\rangle,\ |3, 9, -2, 4\rangle\ \}$. Show that

$$\mathbb{R}\text{-span}(S) = \mathrm{im}(A),\ \text{ where } A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}.$$

KE10HS47. (How kernels and images change) Let $A \in M_{t\times s}(\mathbb{Q})$ and $P \in GL_t(\mathbb{Q})$ and $Q \in GL_s(\mathbb{Q})$. Prove that

$$\begin{aligned} \ker(PA) &= \ker(A), & \ker(AQ) &= Q^{-1} \cdot \ker(A), \\ \mathrm{im}(PA) &= P \cdot \mathrm{im}(A), & \mathrm{im}(AQ) &= \mathrm{im}(A). \end{aligned}$$

KE11HS48. Let $A \in M_{t\times s}(\mathbb{Q})$. Show that $\ker(A)$ is a subspace of $\mathbb{Q}^s$.

KE12HS49. Let $A \in M_{t\times s}(\mathbb{Q})$. Show that $\mathrm{im}(A)$ is a subspace of $\mathbb{Q}^t$.

KE13HS54. Let

$$1_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Find a basis of $\ker(1_2)$.

(b) Find a basis of $\mathrm{im}(1_2)$.

KE14HS50. Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s, t)\}$ and let

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q}).$$

Let

$$A = P1_rQ.$$

Prove that

$$\ker(A) \text{ has basis } \{\text{last } s{-}r \text{ columns of } Q^{-1}\},$$
$$\mathrm{im}(A) \text{ has basis } \{\text{first } r \text{ columns of } P\}.$$

KE15HS51. Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q}).$$

Let

$$A = P1_rQ.$$

Prove that

$$\dim(\mathrm{im}(A)) + \dim(\ker(A)) = (\text{number of columns of } A).$$

KE16HS52. Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. Let $r \in \{1, \ldots, \min(s,t)\}$ and let

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q}).$$

Let

$$A = P1_rQ.$$

Prove that $\dim(\mathrm{im}(A)) = \mathrm{rank}(A)$.

KE17HS55. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 0 \\ -1 & 0 & -2 \\ 1 & 4 & 4 \end{pmatrix}$.

(a) Show that $A = P1_2Q$, where

$$P = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Compute Q^{-1}.

(c) Find a basis of $\ker(A)$.

(d) Find a basis of $\mathrm{im}(A)$.

KE18AQ3. Let

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

and let rowspace(M) be the set of linear combinations of the rows of M. Find a basis of rowspace(M).

KE19AQ4. Let

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

and let colspace(M) be the set of linear combinations of the columns of M. Find a basis of colspace(M).

KE20AQ5. Let

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Find a basis of $\ker(M)$.