15 Vector spaces and linear transformations – Assignment questions

15.1 Subspaces

VS1HS84.

(a) Is the line $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x + 1\}$ a subspace of $\mathbb{R}^2$?

(b) Is the line $L = \{|x, y\rangle \in \mathbb{R}^2 \mid y = 2x\}$ a subspace of $\mathbb{R}^2$?

VS2HS94. Let $T\colon V \to W$ be an $\mathbb{R}$-linear transformation.
Show that $\ker(T) = \{v \in V \mid T(v) = 0\}$ is a subspace of V.

VS3HS95. Let $T\colon V \to W$ be an $\mathbb{R}$-linear transformation.
Show that $\mathrm{im}(T) = \{T(v) \mid v \in V\}$ is a subspace of W.

VS4HS140. Is $W = \{|x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}$ a $\mathbb{R}$-subspace of $\mathbb{R}^3$?

VS5HS141. Is $W = \{a_1 x + a_2 x^2 \mid a_1, a_2 \in \mathbb{R}\}$ a subspace of $\mathbb{R}[x]_{\le 2}$?

VS6HS142. Is the set of real 2×2 matrices whose trace is equal to 0 a subspace of $M_{2\times 2}(\mathbb{R})$?

VS7HS143. Is

$$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) \;\middle|\; ad - bc = 0 \right\} \text{ a subspace of } M_2(\mathbb{R})?.$$

15.2 Linear transformations

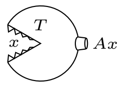
VS8HS85. Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$.
Let $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ be the function given by $T(x) = Ax$ so that

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 \\ 5x_5 + 6x_6 + 7x_7 + 8x_8 \end{pmatrix}.$$

Show that T is a linear transformation.

VS9HS86. Let $t, s \in \mathbb{Z}_{>0}$ and $A \in M_{t\times s}(\mathbb{R})$. Let $T\colon \mathbb{R}^s \to \mathbb{R}^t$ be the function given by

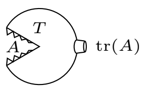
$$T(x) = Ax.$$



Show that T is a linear transformation.

VS10HS87. Let $n \in \mathbb{Z}_{>0}$ and let $T\colon M_n(\mathbb{Q}) \to \mathbb{Q}$ by the function given by

$$T(A) = \mathrm{tr}(A).$$



Show that T is a $\mathbb{Q}$-linear transformation.

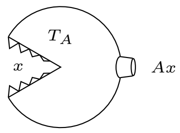
VS11HS88. Let $T\colon M_{2\times 2}(\mathbb{Q}) \to \mathbb{Q}$ be the function given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

Show that T is a linear transformation.

VS12HS130. Let $t, s \in \mathbb{Z}_{>0}$ and $A \in M_{t\times s}(\mathbb{R})$. Let $T_A\colon \mathbb{R}^s \to \mathbb{R}^t$ be the function given by

$$T_A(x) = Ax.$$



Show that T_A is a linear transformation.

VS13HS131. Let $T\colon V \to W$ be a linear transformation. Assume that T has an inverse function $T^{-1}\colon W \to V$. Show that T^{-1} is a linear transformation.

VS14HS144. Is the function $T\colon M_2(\mathbb{R}) \to \mathbb{R}$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \quad \text{a linear transformation?}$$

VS15HS145. Is the function $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T(x_1, x_2, x_3) = (x_2 - 2x_3, 3x_1 + x_3) \qquad \text{a linear transformation?}$$

15.3 Linear independence, span and bases

VS16HS89. Let V be a $\mathbb{Q}$-vector space and let $v_1, v_2, v_3, v_4, v_5 \in V$. Let $S = \{v_1, v_2, v_3, v_4, v_5\}$. Show that $\mathbb{Q}$-span(S) is a subspace of V.

VS17HS90. In $\mathbb{R}[x]_{\le 2}$, is $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$?

VS18HS91. Let S be the subset of $\mathbb{R}[x]_{\le 2}$ given by

$$S = \{1 + 2x, 1 + 5x + 3x^2, x + x^2\}. \qquad \text{Show that } \mathbb{R}\text{-span}(S) = \mathbb{R}[x]_{\le 2}.$$

VS19HS146. In $\mathbb{R}^3$, is $|1, 2, 3\rangle \in \mathbb{R}\text{-span}\{|1, -1, 2\rangle, |-1, 1, 2\rangle\}$?

VS20HS147. In $\mathbb{R}[x]_{\le 2}$, is $1 - 2x - x^2 \in \mathbb{R}\text{-span}\{1 + x + x^2, 3 + x^2\}$?

VS21HS148. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(1, 1, 1), (2, 2, 2), (3, 3, 3)\}. \qquad \text{Determine } \mathbb{R}\text{-span}(S).$$

VS22HS149. Let S be the subset of $\mathbb{R}^2$ given by

$$S = \{|1, -1\rangle, |2, 4\rangle\}. \qquad \text{Show that } \mathbb{R}\text{-span}(S) = \mathbb{R}^2.$$

VS23HS150. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{|1, 2, 0\rangle, |1, 5, 3\rangle, |0, 1, 1\rangle\}. \qquad \text{Show that } \text{span}(S) = \mathbb{R}^3.$$

VS24HS151. Let S be the subset of $\mathbb{R}[x]_{\leq 2}$ given by

$$S = \{1 + x + x^2, x^2\}. \qquad \text{Show that } \mathrm{span}(S) = \mathbb{R}[x]_{\leq 2}.$$

VS25HS152. Let S be the subset of $\mathbb{C}^3$ given by

$$S = \{|2i, -1, 1\rangle, |-6, -3i, 3i\rangle\}. \qquad \text{Is } S\ \mathbb{C}\text{-linearly independent?}$$

VS26HS153. Let B be the subset of $\mathbb{C}^3$ given by

$$B = \{|2i, -1, 1\rangle, |4, 0, 2\rangle\}. \qquad \text{Is } B \text{ linearly independent?}$$

VS27HS154. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(2, 0, 0), (6, 1, 7), (2, -1, 2)\}. \qquad \text{Is } S \text{ linearly independent?}$$

VS28HS155. Let S be the subset of $\mathbb{R}[x]_{\leq 2}$ given by

$$S = \{1 + 2x + 5x^2, 1 + x + x^2, 1 + 2x + 3x^2\}. \quad \text{Is } S \text{ a basis of } \mathbb{R}[x]_{\leq 2}?$$

VS29HS156. Let S be the subset of $M_2(\mathbb{R})$ given by

$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\}. \qquad \text{Is } S \text{ linearly independent?}$$

VS30HS157. Is $S = \{(1, -1), (2, 4)\}$ a basis of $\mathbb{R}^2$?

VS31HS158. Is $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ a basis of $\{A \in M_2(\mathbb{R}) \mid \mathrm{tr}(A) = 0\}$?

VS32HS159. Is

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\} \qquad \text{a basis of } M_2(\mathbb{R})?$$

VS33HS92. Is $S = \{(1, -1), (2, 4)\}$ a basis of $\mathbb{R}^2$?

VS34HS93. Let S be the subset of $M_2(\mathbb{R})$ given by

$$S = \left\{ \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 10 \\ 4 & 2 \end{pmatrix} \right\}. \qquad \text{Is } S \text{ linearly independent?}$$

VS35HS83. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ be given by

$$v_1 = |1, 2, 3\rangle, \quad v_2 = |3, 6, 9\rangle, \quad v_3 = |-1, 0, -2\rangle, \quad v_4 = |1, 4, 4\rangle.$$

(a) Is $\{v_1, v_2, v_3, v_4\}$ linearly independent?
(b) Express v_2 and v_4 as linear combinations of v_1 and v_3.
(c) Is $\{v_1, v_3\}$ linearly independent?

VS36AQ1. Let $\mathbb{R}[x]_{\leq 2} = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$. Let $t \in \mathbb{R}$ and let

$$g_1(x) = 1, \qquad g_2(x) = x + t, \qquad g_3(x) = (x + t)^2.$$

Prove that $\{g_1, g_2, g_3\}$ is a basis of $\mathbb{R}[x]_{\leq 2}$.