11 Matrices and operations – Assignment questions

11.1 Addition, scalar multiplication and multiplication

MO1AS1. Calculate

$$\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \end{pmatrix}.$$

MO2AS2. Calculate

$$\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & -2 & i\sqrt{2} & 0 \\ -1 & -2 & -3 & -4 \end{pmatrix}.$$

MO3AS3. Explain the meaning of 0 in the following expression and compute the sum.

$$0 + \begin{pmatrix} 2 & 3 \\ 6 & 7 \\ -2 & i\sqrt{2} \\ -4.1 & -3 \end{pmatrix}$$

MO4AS4. Explain the meaning of the negative before the first matrix in the following expression and compute the sum.

$$-\begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix} + \begin{pmatrix} 3 & 6 & 2 & 1 \\ 0 & 7 & -9 & 5 \\ 1 & 3.2 & 5 & -1 \end{pmatrix}$$

MO5AS5. Calculate

$$2i \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -5 & 3.2 & i\sqrt{2} & 0 \end{pmatrix}.$$

MO6AS6. Calculate

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \\ -2i \end{pmatrix}.$$

MO7AS7. Calculate

$$\begin{pmatrix} 4 & 2 & 6 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 & 0 & -2i \end{pmatrix}.$$

MO8AS8. Using that $i^2 = -1$, compute

$$\begin{pmatrix} 4 & 2 & 6 & 3 \\ 1 & 2 & -3 & 2i \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \\ -2i & 0 & 1 \end{pmatrix}.$$

MO9AS9. Using $i^2 = -1$, compute

$$\begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 2 \\ 6 & -3 \\ 3 & 2i \end{pmatrix}.$$

MO10AS10. Explain the meaning of 1 and compute the products

$$1 \cdot \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -1 & 2 & 0 & -2i \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \cdot 1.$$

MO11AS11. Compute the following products.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -c & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & c & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

MO12AS12. Compute the following products

$$\begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{8} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{7} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & \frac{8}{7} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{7}{8} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

MO13AS13. Compute the following products.

$$\begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 & 0 & -7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 7 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

MO14AS14. Calculate

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

MO15HS21. Compute

$$\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 \end{pmatrix}.$$

MO16HS22. Compute
$$\frac{1}{3}\begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}.$$

MO17HS23. Let
$$E_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$E_{21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad E_{22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Compute $78E_{11} + 62E_{12} + 91E_{13} + 32E_{21} + 41E_{22} + 24E_{23}$.

MO18HS24. Compute
$$\begin{pmatrix} 2 & 5 & 11 & 13 \end{pmatrix}\begin{pmatrix} 4 \\ 0 \\ 3 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 78 & 62 & 91 & 85 \\ 32 & 41 & 24 & 39 \\ 6 & 99 & 29 & 81 \end{pmatrix}\begin{pmatrix} \frac{2}{100} \\ \frac{85}{100} \\ \frac{1}{100} \\ \frac{12}{100} \end{pmatrix}.$$

MO19HS25. Find $A \in M_2(\mathbb{Q})$ and $B \in M_2(\mathbb{Q})$ such that $AB \neq BA$.

MO20AS15. Calculate the inner product $\langle (4,2,6,3)^t, (-1,2,0,-2i)^t \rangle$ using matrix multiplication.

11.2 Transpose and symmetric, Hermitian, orthogonal and unitary matrices

MO21HS61. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Compute A^T.

MO22HS62. Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$. Prove that A is Hermitian and B is not Hermitian.

MO23HS63. Prove that the matrix $U = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$ is unitary.

MO24HS64. Let $\theta \in \mathbb{R}$. Prove that $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal.

11.3 Determinants and traces

MO25HS71. Let $A = \begin{pmatrix} 2 & 6 & 9 \\ 0 & 3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$. Find the determinant of A by factoring A.

MO26HS72. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the determinant of A by factoring A.

MO27HS73. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$. Compute $\mathrm{tr}(A)$.

MO28HS74. Let $A, B \in M_{n\times n}(\mathbb{Q})$ and let $c \in \mathbb{Q}$. Prove that

$$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B), \qquad \mathrm{tr}(cA) = c\mathrm{tr}(A) \qquad \text{and} \qquad \mathrm{tr}(AB) = \mathrm{tr}(BA).$$

MO29HS75. Let $n \in \mathbb{Z}_{>0}$. Let $A \in M_{n\times n}(\mathbb{Q})$ and let $P \in GL_n(\mathbb{Q})$ so that P is an invertible $n \times n$ matrix. Prove that

$$\det(PAP^{-1}) = \det(A) \qquad \text{and} \qquad \mathrm{tr}(PAP^{-1}) = \mathrm{tr}(A).$$

MO30HS76. Let $A \in M_{n\times n}(\mathbb{Q})$. Prove that

$$\frac{1}{\det(A)} = \det(A^{-1}).$$

MO31HS65. Assume $Q \in O_n(\mathbb{R})$. Using that $\det(AB) = \det(A)\det(B)$ and that $\det(A^T) = \det(A)$,

$$\text{prove that} \qquad \det(Q) \in \{1, -1\}.$$

MO32HS163. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Compute the $(2,3)$ cofactor of A.

(b) Use cofactor expansion along the third row to compute $\det(A)$.

MO33HS164. Using cofactor expansion down the fourth column, compute the determinant of

$$A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 4 \end{pmatrix}.$$