16 Working with linear transformations – Assignment questions

16.1 Matrix of a linear transformation with respect to bases B and C

LT1HS98. Let $v = (1, 5) \in \mathbb{R}^2$.

(a) Find the coordinate vector of v with respect to the basis $S = \{(1, 0), (0, 1)\}$.

(b) Find the coordinate vector of v with respect to the basis $B = \{(2, 1), (-1, 1)\}$.

LT2HS99. Find the coordinate vector of $p = 2+7x-9x^2$ with respect to the basis $B = \{2, \frac{1}{2}x, -3x^2\}$ of $\mathbb{Q}[x]_{\le 2}$.

LT3HS100. The derivative with respect to x is the linear transformation $T\colon \mathbb{R}[x]_{\le 3} \to \mathbb{R}[x]_{\le 2}$ given by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$

(a) Find the matrix of T with respect to the basis $S = \{1, x, x^2, x^3\}$ of $\mathbb{R}[x]_{\le 3}$ and the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$.

(b) Find $\ker(T)$ and $\mathrm{im}(T)$.

LT4HS101. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the function given by $T(x_1, x_2, x_3) = |x_2 - 2x_3, 3x_1 + x_3\rangle$.

(a) Prove that T is a linear transformation.

(b) Find the matrix of T with respect to the basis $S = \{|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle\}$ of $\mathbb{R}^3$ and the basis $B = \{|1, 0\rangle, |0, 1\rangle\}$ of $\mathbb{R}^2$.

LT5HS102. Let $T\colon M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ be the linear transformation given by

$$T(Q) = Q^t.$$

Find the matrix of T with respect to the basis $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$, where E_{ij} is the matrix with 1 in the (i, j) entry and 0 elsewhere.

LT6HS103. Let $T\colon \mathbb{R}[x]_{\le 2} \to \mathbb{R}[x]_{\le 1}$ be the linear transformation given by

$$T(a_0 + a_1x + a_2x^2) = (a_0 + a_2) + a_0x.$$

(a) Find the matrix of T with respect to the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$ and the basis $C = \{1, x\}$ of $\mathbb{R}[x]_{\le 1}$.

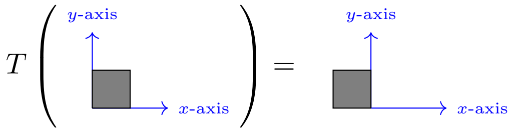
(b) Find the matrix of T with respect to the basis $B = \{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$ and the basis $D = \{2, 3x\}$ of $\mathbb{R}[x]_{\le 1}$.

LT7HS104. Suppose that $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation and that the matrix of T with respect to
the basis $A = \{|1, 0, 0\rangle, |0, 1, 0\rangle, |0, 0, 1\rangle\}$ of $\mathbb{R}^3$ and
the basis $S = \{|1, 0\rangle, |0, 1\rangle\}$ of $\mathbb{R}^2$ is

$$[T]_{SA} = \begin{pmatrix} 5 & 1 & 0 \\ 1 & 5 & -2 \end{pmatrix}.$$

Find the matrix of T with respect to
the basis $B = \{|1,1,0\rangle, |1,-1,0\rangle, |1,-1,-2\rangle\}$ of $\mathbb{R}^3$ and
the basis $C = \{|1,1\rangle, |1,-1\rangle\}$ of $\mathbb{R}^2$.

LT8HS105. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation which is reflection across the y-axis.



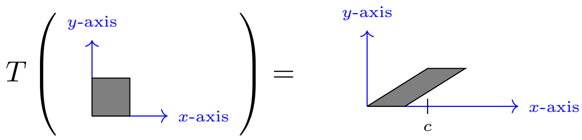
Find the matrix of T with respect to the basis $S = \{|1,0\rangle,\ |0,1\rangle\}$.

LT9HS106. Find the matrix of the linear transformation

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{which is projection onto the } x \text{ axis.}$$

Is T injective? Is T surjective. Is T invertible?

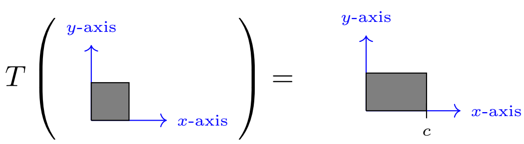
LT10HS107. Let $c \in \mathbb{R}_{>0}$ and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is the shear by a factor of c along the x-axis.



Find the matrix of T with respect to the basis $S = \{|1,0\rangle, |0,1\rangle\}$.

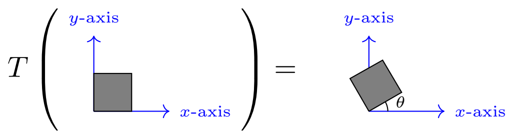
LT11HS108. Find the image of $(x,y) \in \mathbb{R}^2$ after a shear along the x-axis with $c = 1$ followed by a reflection across the y-axis.

LT12HS109. Let $c \in \mathbb{R}_{>0}$ and let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is stretching of the x-axis by a factor of c.



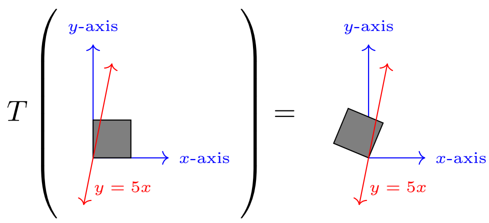
Find the matrix of T with respect to the basis $B = \{(1,0), (0,1)\}$.

LT13HS110. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is rotation by θ (about the origin counterclockwise).



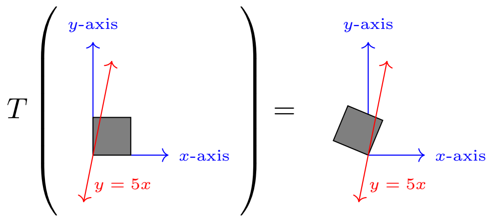
Find the matrix of T with respect to the basis $S = \{(1,0), (0,1)\}$.

LT14HS111. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection in the line $y = 5x$.



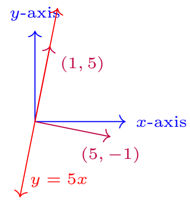
Identify two lines through the origin that are invariant under T and find the image of the direction vectors for each of these lines.

LT15HS112. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which is reflection in the line $y = 5x$.



Let $S = \{(1,0),(0,1)\}$ and let

$$B = \{(1,5),(5,-1)\},$$



so that the first vector in B is a vector in the direction of the line $y = 5x$ and the second vector in B is a vector perpendicular to $y = 5x$.

(a) Compute $[T]_{BB}$, $[I]_{SB}$, $[I]_{BS}$, and $[T]_{SS}$.

(b) Compute $T\begin{pmatrix}1\\0\end{pmatrix}$ and $T\begin{pmatrix}0\\1\end{pmatrix}$.

LT16HS113. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x,y) = (3x - y, -x + 3y).$$

and let B and S be the bases of $\mathbb{R}^2$ given by

$$B = \{(1,1),(-1,1)\} \qquad \text{and} \qquad S = \{(1,0),(0,1)\}.$$

Let u and v be the vectors in $\mathbb{R}^2$ determined by

$$[u]_B = \begin{pmatrix}1\\1\end{pmatrix} \qquad \text{and} \qquad [v]_S = \begin{pmatrix}0\\2\end{pmatrix}.$$

(a) Compute $[I]_{BS}$, and $[I]_{SB}$

(b) Compute $[u]_S$ and $[v]_B$.

(c) Compute $[T]_{SS}$ and $[T]_{BB}$.

16.2 Kernels and images of linear transformations

LT17HS96. Let $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x, y, z) = (2x - y, y + z).$$

Find bases for $\ker(T)$ and $\mathrm{im}(T)$ and verify the rank-nullity theorem.

LT18HS97. Let $T\colon \mathbb{R}[x]_{\le 2} \to \mathbb{R}[x]_{\le 1}$ be the linear transformation given by

$$T(a_0 + a_1x + a_2x^2) = (a_0 - a_1 + a_2)(1 + 2x).$$

(a) Find bases for $\ker(T)$ and $\mathrm{im}(T)$.
(b) Is T injective?
(c) Is T surjective?