20 Learning to do proofs – Assignment questions

PF1HS132. Prove the following theorem.

Theorem 20.1. ***(Exchange Theorem)*** *Let V be an $\mathbb{F}$-vector space. Let $B = \{v_1, \ldots, v_k\}$ be a basis of V and let $D = \{d_1, \ldots, d_\ell\}$ be another basis of V. Then there exists $d_{i_1} \in D$ such that*

$$\{d_{i_1}, b_2, b_3, \ldots, b_k\} \quad \text{is a basis of } V.$$

PF2HS134. Prove the following theorem.

Theorem 20.2. ***(Dimension Theorem)*** *Let V be an $\mathbb{F}$-vector space. Any two bases of V have the same number of elements.*

The following provides an example of a spanning set that is not minimal, and another spanning set for the same subspace that is minimal.

PF3Example V14. Let S be the subset of $\mathbb{R}^3$ given by

$$S = \{(1,1,1), (2,2,2), (3,3,3)\}. \qquad \text{Determine } \mathbb{R}\text{-span}(S).$$

PF4. Prove the following proposition.

Proposition 20.3. ***(Span is a subspace)*** *Let V be a $\mathbb{R}$-vector space. Let $B = \{b_1, \ldots, b_k\}$ be a subset of V. Then $\mathbb{R}$-span(B) is a subspace of V.*

PF5HS134. Prove the following proposition.

Proposition 20.4. *Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$.*
If $\ker(A) = 0$ then the columns of A are linearly independent.

PF6HS134. Prove the following proposition.

Proposition 20.5. *Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$. Then*

$$im(A) = \mathbb{R}\text{-}span\{\text{columns of } A\}.$$

PF7HS133. Prove the following proposition.

Proposition 20.6. *Let $A \in M_{t\times s}(\mathbb{Q})$. Then $\ker(A)$ is a subspace of $\mathbb{Q}^s$.*

PF8HS133. Prove the following proposition.

Proposition 20.7. *Let $A \in M_{t\times s}(\mathbb{Q})$. Then $im(A)$ is a subspace of $\mathbb{Q}^t$.*

PF9Example A5. Let $T: V \to W$ be an $\mathbb{R}$-linear transformation.
Show that $\ker(T) = \{v \in V \mid T(v) = 0\}$ is a subspace of V.

PF10Example A6. Let $T: V \to W$ be an $\mathbb{R}$-linear transformation.
Show that $\text{im}(T) = \{T(v) \mid v \in V\}$ is a subspace of W.

PF11HS133. Prove the following theorem.

Theorem 20.8. ***(Basis Minimax Theorem)*** *Let V be an $\mathbb{F}$-vector space and let B be a subset of V. The following are equivalent:*

(a) B is a basis of V.

(b) B is a minimal spanning set of V.

(c) B is a maximal linearly independent set of V.

PF12HS134. Prove the following proposition.

Proposition 20.9. *Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$. Let $a_1, \ldots, a_s$ be the columns of A. Then*

$$im(A) = span\{a_1, \ldots, a_s\}.$$

PF13HS137. Prove the following theorem.

Theorem 20.10. ***(Invertible matrices are square)*** *Let $s, t \in \mathbb{Z}_{>0}$ and let $A \in M_{t\times s}(\mathbb{R})$. Suppose there exists*

$$P \in M_{s\times t}(\mathbb{R}) \quad \text{be such that} \quad PA = 1.$$

Suppose there exists

$$Q \in M_{s\times t}(\mathbb{R}) \quad \text{be such that} \quad AQ = 1.$$

Then

(a) $\ker(A) = 0$.

(b) $im(A) = \mathbb{R}^t$.

(c) The set of columns of A is a basis of $\mathbb{Q}^t$.

(d) $s = t$.

(e) $P = Q$.