12 Inverses, normal form and rank – Assignment questions

12.1 Inverses

IF1HS26. Let $c \in \mathbb{Q}$. Find the inverse of the matrix $\begin{pmatrix} c & 1 \\ 1 & 0 \end{pmatrix}$.

IF2HS27. Let $c \in \mathbb{Q}$. FInd the inverses of the matrices

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

IF3HS28. Let $d \in \mathbb{Q}$ and assume $d \neq 0$. Find the inverses of the matrices

$$h_1(d) = \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ h_2(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ h_3(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

IF4HS29. Let $c \in \mathbb{Q}$. Find the inverses of the matrices

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

IF5HS30. Find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

IF6HS31. Find the inverse of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

IF7HS32. Let $A, B \in GL_n(\mathbb{Q})$. Prove that $AB \in GL_n(\mathbb{Q})$ by finding $(AB)^{-1}$.

12.2 Normal form

IF8HS33. By multiplying out the matrices on the left hand side check that

$$s_4(1)s_3(2)s_2(3)s_1(4) \cdot s_4(5)s_3(6)s_2(7) \cdot s_4(8)s_3(9) \cdot s_4(10) = \begin{pmatrix} 4 & 7 & 9 & 10 & 1 \\ 3 & 6 & 8 & 1 & 0 \\ 2 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

IF9HS34. By multiplying out the matrices on the left hand side check that

$$h_1(15)h_2(-3)h_3(76)h_4(-19)h_5(2) = \begin{pmatrix} 15 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 76 & 0 & 0 \\ 0 & 0 & 0 & -19 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

IF10HS35. (a) Check that

$$\begin{pmatrix} 1 & -10 & 71 & -302 & 186 \\ 0 & 1 & -8 & 34 & -21 \\ 0 & 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 10 & 9 & 7 & 4 \\ 0 & 1 & 8 & 6 & 3 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) By multiplying out the matrices on the left hand side check that

$$\begin{array}{l} x_{45}(1)x_{35}(2)x_{25}(3) \\ \cdot x_{15}(4)x_{34}(5)x_{24}(6) \\ \cdot x_{14}(7)x_{23}(8)x_{13}(9) \\ \cdot x_{12}(10) \end{array} = \begin{pmatrix} 1 & 10 & 9 & 7 & 4 \\ 0 & 1 & 8 & 6 & 3 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) By multiplying out the matrices on the left hand side check that

$$\begin{array}{l} x_{12}(10)^{-1}x_{13}(9)^{-1}x_{23}(8)^{-1} \\ \cdot x_{14}(7)^{-1}x_{24}(6)^{-1}x_{34}(5)^{-1} \\ \cdot x_{15}(4)^{-1}x_{25}(3)^{-1}x_{35}(2)^{-1} \\ \cdot x_{45}(1)^{-1} \end{array} = \begin{pmatrix} 1 & -10 & 71 & -302 & 186 \\ 0 & 1 & -8 & 34 & -21 \\ 0 & 0 & 1 & -5 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

IF11HS36. (a) Check that

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(b) By multiplying out the matrices on the left hand side check that

$$s_1(\tfrac{1}{3})h_1(3)h_2(\tfrac{2}{3})x_{12}(\tfrac{4}{3}) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

(c) By multiplying out the matrices on the left hand side check that

$$x_{12}(\tfrac{4}{3})^{-1}h_2(\tfrac{2}{3})^{-1}h_1(3)^{-1}s_1(\tfrac{1}{3})^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

IF12HS37. (a) Check that

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) By multiplying out the matrices on the left hand side check that

$$s_1(-1)s_2(1)h_1(-1)h_3(-1)x_{23}(3)x_{13}(-1)x_{12}(1) = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

(c) By multiplying out the matrices on the left hand side check that

$$\begin{matrix} x_{12}(1)^{-1}x_{13}(-1)^{-1}x_{23}(3)^{-1} \\ \cdot h_3(-1)^{-1}h_1(-1)^{-1}s_2(1)^{-1}s_1(-1)^{-1} \end{matrix} = \begin{pmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}.$$

IF13HS38. Let $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{pmatrix} \in M_{3\times 4}(\mathbb{Q})$.
By multiplying out the matrices on the left hand side check that

$$s_2(0)s_1(1)s_2(2)x_{12}(-3)\cdot 1_2 \cdot x_{23}(1)x_{14}(5)x_{24}(-2) = A.$$

Determine $\mathrm{rank}(A)$.

IF14AQ0. Find the normal form of

$$A = \begin{pmatrix} -3 & -36 & -39 & -42 & -46 & -84 \\ -9 & -108 & -\frac{233}{2} & -114 & \frac{293}{2} & -167 \\ 3 & 36 & 39 & 42 & 45 & 48 \\ 0 & 0 & \frac{1}{4} & 6 & \frac{25}{4} & \frac{13}{2} \end{pmatrix}.$$

This means, find $P \in GL_4(\mathbb{Q})$, $Q \in GL_6(\mathbb{Q})$ and $r \in \{0,1,2,3,4\}$ such that $A = P1_rQ$ and find R in reduced row echelon form such that $A = PR$.

IF15AQ2. Find the normal form of

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 & 2 \\ 1 & 4 & 0 & 1 & 2 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

This means find $P \in GL_4(\mathbb{Q})$, $Q \in GL_5(\mathbb{Q})$ and $r \in \{0,1,2,3,4\}$ such that $A = P1_rQ$ and find R in reduced row echelon form such that $A = PR$.

IF16HS53. Let $A \in M_{t\times s}(\mathbb{Q})$. Show that there exist $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$ and $r \in \{1,\ldots,\min(s,t)\}$ such that

$$A = P1_rQ.$$

where $1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t\times s}(\mathbb{Q})$.

IF17AQ6. Find the normal form of

$$A = \begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

This means find $P \in GL_4(\mathbb{Q})$, $Q \in GL_4(\mathbb{Q})$ and $r \in \{0,1,2,3,4\}$ such that $A = P1_rQ$ and find R in reduced row echelon form such that $A = PR$.

IF18AQ7. (Row reducing a generic 2×1 matrix) Show that if $a, b \in \mathbb{Q}$ and $b \neq 0$ then $\begin{pmatrix} a \\ b \end{pmatrix} = s_1(\frac{a}{b})\begin{pmatrix} b \\ 0 \end{pmatrix}$.

IF19AQ8. (Row reducing a generic 3×3 matrix) Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be an element of $M_{3\times 3}(\mathbb{Q})$ and assume that $a_{31} \neq 0$ and $a_{22} - \frac{a_{32}a_{21}}{a_{31}} \neq 0$. Show that

$$A = s_2\Big(\frac{a_{21}}{a_{31}}\Big)s_1\Big(\frac{a_{11}}{a_{31}}\Big)s_2\left(\frac{a_{12} - \frac{a_{32}a_{11}}{a_{31}}}{a_{22} - \frac{a_{32}a_{21}}{a_{31}}}\right)\begin{pmatrix} a_{31} & a_{32} & a_{33} \\ 0 & a_{22} - \frac{a_{32}a_{21}}{a_{31}} & a_{23} - \frac{a_{33}a_{21}}{a_{31}} \\ 0 & 0 & a_{13} - \frac{a_{33}a_{11}}{a_{31}} - \frac{\left(a_{23} - \frac{a_{33}a_{21}}{a_{31}}\right)\left(a_{12} - \frac{a_{32}a_{11}}{a_{31}}\right)}{a_{22} - \frac{a_{32}a_{21}}{a_{31}}} \end{pmatrix}.$$