17 Inner products – Assignment questions

17.1 Standard inner products and lengths, distances and angles between vectors

IP1HS114. Let $u = |1+i, 1-i\rangle$ and $v = |i, 1\rangle$ in $\mathbb{C}^2$ with the standard inner product.
Compute $\langle u|v\rangle$, $\langle v|u\rangle$, $\langle u|u\rangle$, $\langle u|u\rangle$, $d(u,v)$, $\cos(\theta(u,v))$ and $\theta(u,v)$.

IP2HS115. Let $u = 1$ and $v = x$ in $\mathbb{R}[x]_{\le 2}$ with the standard inner product.
Compute $\langle u|u\rangle$, $\langle v|v\rangle$, $\langle u|v\rangle$, $d(u,v)$, $\cos(\theta(u,v))$, and $\theta(u,v)$.

IP3AQ12. Let $\langle , \rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\langle (u_1,u_2),(v_1,v_2)\rangle = u_1v_1 + 2u_2v_2 = \begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Let $u = |3,1\rangle$ and $v = |-2,3\rangle$. Find $\|u\|$, $d(u,v)$ and $\theta(u,v)$.

17.2 Gram matrices and positive definiteness

IP4HS116. Let $A \in M_{n\times n}(\mathbb{C})$ satisifying $A = \bar{A}^t$ and let $\langle , \rangle \colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ be given by

$$\langle (u_1,\ldots,u_n),(v_1,\ldots,v_n)\rangle = (u_1,\ldots,u_n)A \begin{pmatrix} \overline{v_1} \\ \vdots \\ \overline{v_n} \end{pmatrix} = u^t A\bar{v},$$

Prove that $\langle , \rangle$ satisfies all the properties of an inner product, except perhaps the positive definiteness.

IP5HS117. Is the map $\langle , \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$\langle (u_1,u_2,u_3),(v_1,v_2,v_3)\rangle = u_1v_1 - u_2v_2 + u_3v_3 \qquad \text{positive definite?}$$

IP6HS118. Is the map $\langle , \rangle \colon \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$ given by

$$\langle (u_1,u_2),(v_1,v_2)\rangle = iu_1\overline{v_1} - iu_2\overline{v_2} \qquad \text{positive definite?}$$

IP7HS119. Is the map $\langle , \rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\langle (u_1,u_2),(v_1,v_2)\rangle = 2u_1v_1 - 2u_1v_2 - 2u_2v_1 + u_2v_2 \qquad \text{positive definite?}$$

IP8HS120. Is the map $\langle , \rangle \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$\langle (u_1,u_2),(v_1,v_2)\rangle = u_1v_1 + 2u_2v_2 \qquad \text{positive definite?}$$

IP9HS121. (Pythagorean theorem) Let V be an $\mathbb{F}$-vector space with an inner product. Let $u,v \in V$ and suppose that u and v are orthogonal. Prove that $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.

IP10HS122. Let V be a $\mathbb{F}$-vector space with an inner product $\langle , \rangle \colon V \times V \to \mathbb{F}$.
Let $B = \{b_1,\ldots,b_n\}$ be a basis of V and let A be the Gram matrix of $\langle , \rangle$ with respect to the basis B. Let $u,v \in V$. Prove that

$$\langle u,v\rangle = [u]_B^t A[\bar{v}]_B.$$

IP11HS123. Let $\langle\ ,\ \rangle \colon \mathbb{R}[x]_{\le 2} \times \mathbb{R}[x]_{\le 2} \to \mathbb{R}$ be the standard inner product on the $\mathbb{R}$-vector space $\mathbb{R}[x]_{\le 2}$.

(a) Let $u = 7 + 3x + 2x^2$ and $v = 5 + x^2$. Calculate $\langle u|v\rangle$.

(b) Let A be the Gram matrix of $\langle\,,\,\rangle$ with respect to the basis $B = \{1, x, x^2\}$. Calculate A.

(c) Calculate $[u]_B$ and $[v]_B$ and $[u]_B^T A[v]_B$.

IP12HS66. Let $u, v \in \mathbb{R}^n$ and let $Q \in O_n(\mathbb{R})$. Prove that $\langle Qu|Qv\rangle = \langle u|v\rangle$.

17.3 Orthogonality and orthogonalization

IP13HS124. Let V be an $\mathbb{F}$-vector space with an inner product $\langle,\rangle\colon V \times V \to \mathbb{F}$. Assume $B = (b_1, \ldots, b_n)$ is an ordered orthonormal basis of V and $x \in V$. Prove that

$$x = \langle x, b_1\rangle b_1 + \cdots + \langle x, b_n\rangle b_n.$$

IP14HS125. Let $\langle,\rangle\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be given by

$$\langle(u_1, u_2, u_3), (v_1, v_2, v_3)\rangle = u_1v_1 + 2u_2v_2 + u_3v_3.$$

(a) Prove that $S = \{(1, 1, 1), (1 - 1, 1), (1, 0 - 1)\}$ is an orthogonal sequence in $\mathbb{R}^3$ with respect to $\langle,\rangle$.

(b) Let

$$b_1 = \frac{1}{\|u\|}u, \qquad \text{where} \quad u = (1, 1, 1),$$
$$b_2 = \frac{1}{\|v\|}v, \qquad \text{where} \quad v = (1, -1, 1),$$
$$b_3 = \frac{1}{\|w\|}w, \qquad \text{where} \quad w = (1, 0, 01).$$

Show that $\{b_1, b_2, b_3\}$ is an orthonormal sequence in $\mathbb{R}^3$ with respect to $\langle,\rangle$.

(c) Let $x = |1, 1, -1\rangle$ and $c_2, c_2, c_3 \in \mathbb{R}$ such that

$$x = c_1b_1 + c_2b_2 + c_3b_3.$$

Compute c_1, c_2, c_3.

IP15HS126. Let $V = \mathbb{R}^n$ and let $u \in V$ with $u \neq 0$. Let

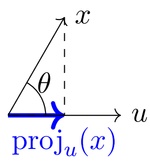
$$W = \mathbb{R}\text{-span}\{u\} = \{au \mid a \in \mathbb{R}\}.$$

(a) Show that W is a subspace of V.

(b) Find an orthonormal basis of W.

(c) Let $x \in V$. Prove that

$$\mathrm{proj}_W(x) = \mathrm{proj}_u(x).$$



IP16HS127. Let $V = \mathbb{R}^3$ and let $W = \{|x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}$.

(a) Show that the set
$$\{b_1, b_2\} = \left\{\tfrac{1}{\sqrt{2}}|1, -1, 0\rangle, \tfrac{1}{\sqrt{6}}|1, 1, -2\rangle\right\}$$
is an orthonormal basis of W with respect to the standard inner product on $\mathbb{R}^3$.

(b) Let $x = |1, 2, 3\rangle$. Compute $\mathrm{proj}_W(x)$.

(c) Compute the shortest distance from x to W.

IP17HS128. Let $V = \mathbb{R}^3$ with the standard inner product. Let $S = \{v_1, v_2, v_3\}$ with
$$v_1 = |1, 1, 1\rangle, \quad v_2 = |0, 1, 1\rangle, \quad v_1 = |0, 0, 1\rangle.$$
Apply the Gram-Schmidt process to convert S into an orthonormal basis B.

IP18HS129. Let V be a $\mathbb{C}$-vector space with inner product $\langle , \rangle \colon V \times V \to \mathbb{C}$.
Let $B = \{b_1, \ldots, b_k\}$ be an orthogonal set in V. Prove that B is linearly independent.

IP19AQ11. (Gram-Schmidt with polynomials) Assume that $\mathbb{R}[x]_{\le 2}$ has the inner product
$$\langle p, q\rangle = \int_0^1 p(x)q(x)dx.$$
Starting with the basis $\{1, x, x^2\}$ of $\mathbb{R}[x]_{\le 2}$ find an orthonormal basis of $\mathbb{R}[x]_{\le 2}$.

IP20AQ14. (Gram-Schmidt) Use the standard inner product on $\mathbb{R}^3$ and apply Gram-Schmidt process to the set
$$S = \{|1, 1, 1\rangle, |0, 1, 1\rangle, |0, 0, 1\rangle\}.$$
Analyze the process to show that
$$P^tAP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{where} \quad P = h_1\left(\tfrac{1}{\sqrt{3}}\right)x_{12}\left(\tfrac{-2}{\sqrt{3}}\right)h_2\left(\tfrac{3}{\sqrt{2}}\right)x_{23}\left(\tfrac{-1}{\sqrt{6}}\right)x_{13}\left(\tfrac{-1}{\sqrt{3}}\right)h_3\left(\tfrac{1}{\sqrt{2}}\right).$$