18 Geometry in $\mathbb{R}^n$ – Assignment questions

GE1HS1. Let

$$e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Compute $3e_1 + 5e_2 + (-2)e_3 + 0e_4$.

GE2HS2. Let $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that

$$\langle y, x\rangle = \langle x, y\rangle, \qquad \langle x, y\rangle = x^T y, \qquad \|x\| = \sqrt{\langle x, x\rangle}, \qquad d(x, y) = \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}.$$

GE3HS3. Let $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that

$$\langle y, x\rangle = \langle x, y\rangle, \qquad \langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle, \qquad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle,$$

$$\langle x, cy\rangle = c\langle x, y\rangle, \qquad \langle cx, y\rangle = c\langle x, y\rangle, \qquad \|cx\| = |c| \cdot \|x\|.$$

GE4HS4. Let $x, y \in \mathbb{R}^n$. Prove that

$$|\langle x, y\rangle| \le \|x\| \cdot \|y\|.$$

GE5HS5. Let $x, y \in \mathbb{R}^n$. Prove that

$$\|x + y\| \le \|x\| + \|y\|$$

GE6HS6. Let $u = |1, 3, 1, 2\rangle$ and $v = |2, 1, -1, 3\rangle$ in $\mathbb{R}^4$. Compute $u - v$ and $d(u, v)$.

GE7HS7. Let $u = |0, 2, 2, -1\rangle$ and $v = |-1, 1, 1, -1\rangle$ in $\mathbb{R}^4$. Compute $\langle u, v\rangle$, $\|u\|$ and $\|v\|$ and check that $|\langle u, v\rangle| \le \|u\| \cdot \|v\|$.

GE8HS8. Let $u = (2, -1, -2)$ and $v = (2, 1, 3)$. Find vectors v_1 and v_2 such that

$$v = v_1 + v_2$$

where v_1 is parallel to u and v_2 is perpendicular to u.

GE9HS9. Determine the vector, parametric and Cartesian equations of the line through the points $P = (-1, 2, 3)$ and $Q = (4, -2, 5)$.

GE10HS10. Find a vector equation of the 'friendly' line through the point $(2, 0, 1)$ that is parallel to the 'enemy' line

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-6}{2}.$$

Does the point $(0, 4, -3)$ lie on the 'friendly' line?

GE11HS11. Find the vector equation for the plane in $\mathbb{R}^3$ containing the points $P = |1, 0, 2\rangle$ and $Q = |1, 2, 3\rangle$ and $R = |4, 5, 6\rangle$.

GE12HS12. Where does the line

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$

intersect the plane $3x + 2y + z = 20$?

GE13HS13. Find a vector form for the line of intersection of the two planes $x + 3y + 2z = 6$ and $3x + 2y + z = 11$.

GE14HS14. Let $\hat{\imath}, \hat{\jmath}, \hat{k} \in \mathbb{R}^3$ be given by

$$\hat{\imath} = |1, 0, 0\rangle, \qquad \hat{\jmath} = |0, 1, 0\rangle, \qquad \hat{k} = |0, 0, 1\rangle.$$

Compute $5\hat{\imath} + (-1)\hat{\jmath} + (-4)\hat{k}$.

GE15HS15. Let $u = |u_1, u_2, u_3\rangle, v = |v_1, v_2, v_3\rangle, w = |w_1, w_2, w_3\rangle$ in $\mathbb{R}^3$. Prove that

$$u \times v = \det\begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} \hat{\imath} - \det\begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} \hat{\jmath} + \det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \hat{k}, \qquad \text{and}$$

$$\langle u, v \times w\rangle = \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

GE16HS16. Let $u = |u_1, u_2, u_3\rangle, v = |v_1, v_2, v_3\rangle, w = |w_1, w_2, w_3\rangle$ In $\mathbb{R}^3$. Prove that

$$\langle v, v \times w\rangle = 0 \qquad \text{and} \qquad \langle w, v \times w\rangle = 0.$$

Conclude that $v \times w$ is perpendicular to both v and w.

GE17H17. Find a vector perpendicular to both $|1, 1, 1\rangle$ and $|1, -1, -2\rangle$.

GE18HS18. Find the volume of the parallelipiped with adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$, $\overrightarrow{PS}$, where

$$P = |2, 0, -1\rangle, \qquad Q = |4, 1, 0\rangle, \qquad R = |3, -1, 1\rangle \qquad \text{and} \qquad S = |2, -2, 2\rangle.$$

GE19HS19. Find the area of the triangle in $\mathbb{R}^3$ with vertices $|2, -5, 4\rangle$, $|3, -4, 5\rangle$ and $|3, -6, 2\rangle$.

GE20HS20. Find the Cartesian equation of the plane with vector form

$$|x, y, z\rangle = s\,|1, -1, 0\rangle + t\,|2, 0, 1\rangle + |-1, 1, 1\rangle, \text{ with } s, t \in \mathbb{R}.$$