14 Eigenvectors and diagonalization – Assignment questions

14.1 Eigenvalues and eigenvectors

EV1HS56. Let $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find $P, D \in GL_2(\mathbb{Q})$ such that D is diagonal and $A = PDP^{-1}$.

EV2HS57. Let $A = \begin{pmatrix} 2 & -3 & 6 \\ 0 & 5 & -6 \\ 0 & 1 & 0 \end{pmatrix}$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find $P, D \in GL_2(\mathbb{Q})$ such that D is diagonal and $A = PDP^{-1}$.

EV3HS58. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ viewed as an element of $M_{2\times 2}(\mathbb{R})$. Show that A does not have any eigenvalues or eigenvectors. Conclude that A is not diagonalizable as an element of $M_{2\times 2}(\mathbb{R})$.

EV4HS59. Let $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ viewed as an element of $M_{2\times 2}(\mathbb{C})$.

(a) Find the eigenvalues and eigenvectors of A.

(b) Find $P, D \in GL_2(\mathbb{C})$ such that D is diagonal and $A = PDP^{-1}$.

Conclude that A is diagonalizable as an element of $M_{2\times 2}(\mathbb{C})$.

EV5HS60. Find a matrix $A \in M_{2\times 2}(\mathbb{C})$ that does not have 2 linearly independent eigenvectors. Conclude that A is not diagonalizable.

14.2 Diagonalization

EV6HS79. Let $D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Compute D^{10}.

EV7HS80. Assume $A = PDP^{-1}$. Prove that if $k \in \mathbb{Z}_{>0}$ then $A^k = PD^kP^{-1}$.

EV8HS80b. Assume $A = PDP^{-1}$ and D is invertible. Prove that if $k \in \mathbb{Z}_{>0}$ then $A^{-k} = PD^{-k}P^{-1}$.

EV9HS81. Let

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}.$$

(a) Compute P^{-1} and prove that $A = PDP^{-1}$.

(b) Compute A^k for $k \in \mathbb{Z}$.

EV10HS82. Let
$$x_n = \begin{pmatrix} r_n \\ p_n \\ w_n \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} \frac12 & \frac14 & 0 \\ \frac12 & \frac12 & \frac12 \\ 0 & \frac14 & \frac12 \end{pmatrix}.$$

(a) Show that
$$\text{if} \quad P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{then} \quad P^{-1} = \begin{pmatrix} \frac14 & \frac14 & \frac14 \\ -\frac12 & 0 & \frac12 \\ \frac14 & -\frac14 & \frac14 \end{pmatrix}.$$

(b) Show that $T = PDP^{-1}$.

(c) Let $r_0, p_0, w_0 \in \mathbb{R}$ and let
$$x_0 = \begin{pmatrix} r_0 \\ p_0 \\ w_0 \end{pmatrix}. \qquad \text{Compute } \lim_{n\to\infty} T^n x_0.$$

(d) Explain how this is used in applications to genetics.

14.3 Diagonalization with symmetric, hermitian, orthogonal and unitary matrices

EV11HS67. Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Find an orthogonal matrix $Q \in O_2(\mathbb{R})$ and a diagonal matrix $D \in M_{2\times 2}(\mathbb{R})$ such that $A = QDQ^T$.

EV12HS68. Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Find an unitary matrix $Q \in U_2(\mathbb{C})$ and a diagonal matrix $D \in M_{2\times 2}(\mathbb{C})$ such that $A = QDQ^T$.

EV13HS69. Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Find $U \in O_2(\mathbb{R})$, $V \in O_3(\mathbb{R})$ and $S \in M_{2\times 3}(\mathbb{R})$ such that S is diagonal and $A = USV^T$.

EV14HS70. Let $A = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. Find $U \in O_2(\mathbb{R})$, $V \in O_2(\mathbb{R})$ and $S \in M_{2\times 2}(\mathbb{R})$ such that S is diagonal and $A = USV^T$.