MAST10007 Linear Algebra

THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

Summer Term, 2026

Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Lecture 9: Equations of lines and planes in $\mathbb{R}^3$

Definition

The *line in $\mathbb{R}^3$ with direction $\mathbf{v} = \langle a, b, c\rangle$ going through the point $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$* is

$$r_0 + \mathbb{R}\mathbf{v} = \{r_0 + t\mathbf{v} \mid t \in \mathbb{R}\} \quad \textit{PICTURE}.$$

Definition

The *plane in $\mathbb{R}^3$ spanned in directions $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ going through the point $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$* is

$$r_0 + \mathbb{R}\mathbf{u} + \mathbb{R}v = \{r_0 + s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\} \quad \textit{PICTURE}.$$

Equations of lines in $\mathbb{R}^3$

Definition

The *line in $\mathbb{R}^3$ with direction $\mathbf{v} = |a, b, c\rangle$ going through the point $\mathbf{r}_0 = |x_0, y_0, z_0\rangle$* is

$$r_0 + \mathbb{R}\mathbf{v} = \{r_0 + t\mathbf{v} \mid t \in \mathbb{R}\} \quad \textit{PICTURE}.$$

The points in the line are the $|x, y, z\rangle$ in $\mathbb{R}^3$ such that

$$(x, y, z) = (x_0, y_0, z_0) + t(a, b, c), \quad \text{with } t \in \mathbb{R}, \qquad \textit{(vector equation)}$$

or

$$\begin{aligned} x &= x_0 + ta, \\ y &= y_0 + tb, \\ z &= z_0 + tc, \end{aligned} \qquad \text{with } t \in \mathbb{R}, \qquad \textit{(parametric equation)}$$

Solving for t gives that the points on the line are the $|x, y, z\rangle$ in $\mathbb{R}^3$ which satisfy the equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \qquad \textit{(Cartesian form)}$$

Equations of planes in $\mathbb{R}^3$

Definition

The *plane in $\mathbb{R}^3$ spanned in directions $\mathbf{u} = |u_1, u_2, u_3\rangle$ and $\mathbf{v} = |v_1, v_2, v_3\rangle$ going through the point $\mathbf{r}_0 = |x_0, y_0, z_0\rangle$* is

$$r_0 + \mathbb{R}\mathbf{u} + \mathbb{R}v = \{r_0 + s\mathbf{u} + t\mathbf{v} \mid s, t \in \mathbb{R}\} \quad PICTURE.$$

The points in the line are the $|x, y, z\rangle$ in $\mathbb{R}^3$ such that

$$\begin{aligned} x &= x_0 + su_1 + tv_1, \\ y &= y_0 + su_2 + tv_2, \\ z &= z_0 + su_3 + tv_3, \end{aligned} \qquad \text{with } s, t \in \mathbb{R}. \qquad \textit{(parametric equation)}$$

The *vector equation* is

$$(x, y, z) = (x_0, y_0, z_0) + s(u_1, u_2, u_3) + t(v_1, v_2, v_3), \quad \text{with } s, t \in \mathbb{R}.$$

Let $\mathbf{n} = |a, b, c\rangle$ be such that n is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. In other words, $\mathbf{n}$ is a vector perpendicular to the plane. Then

$$\begin{aligned}\langle \mathbf{n}|x, y, z\rangle &= \langle n, \mathbf{r}_0 + s\mathbf{u} + t\mathbf{v}\rangle = \langle \mathbf{n}, \mathbf{r}_0\rangle + s\langle \mathbf{n}, \mathbf{u}\rangle + t\langle \mathbf{n}, \mathbf{v}\rangle \\ &= \langle \mathbf{n}, \mathbf{r}_0\rangle + s \cdot 0 + t \cdot 0 = \langle \mathbf{n}, \mathbf{r}_0\rangle,\end{aligned}$$

and since $\langle \mathbf{n}|x, y, z\rangle = \langle a, b, c|x, y, z\rangle = ax + by + cz$ then the plane is the set of $|x, y, z \in \mathbb{R}^3$ such that

$$ax + by + cz = \langle \mathbf{r}_0, \mathbf{n}\rangle. \qquad \textit{(Cartesian form)}$$

Example E8. Determine the vector, parametric and Cartesian equations of the line through the points $P = (-1, 2, 3)$ and $Q = (4, -2, 5)$.

Since the direction of the line is

$$Q - P = |4, -2, 5\rangle - |-1, 2, 3\rangle = |5, -4, 2\rangle$$

and

$$P = |-1, 2, 3\rangle \text{ is a point on the line}$$

then the line is the set of points in $\mathbb{R}^3$ given by

$$\{\, |-1, 2, 3\rangle + \; t \cdot |5, -4, 2\rangle \mid t \in \mathbb{R}\}.$$

Parametric equations for the line are

$$\begin{aligned} x &= -1 + 5t, \\ y &= 2 - 4t, \\ z &= 3 + 2t, \end{aligned} \qquad \text{with } t \in \mathbb{R}.$$

Solving for t, the Cartesian equation of the line is

$$\frac{x+1}{5} = \frac{y-2}{-4} = \frac{z-3}{2}.$$

Example E9. Find a vector equation of the "friendly" line through the point $(2, 0, 1)$ that is parallel to the "enemy" line

$$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-6}{2}.$$

Does the point $(0, 4, -3)$ line on the "friendly" line?

Letting

$$t = \frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-6}{2}$$

gives

$$\begin{aligned} x &= 1 + t, \\ y &= -2 - 2t, \quad \text{with } t \in \mathbb{R}, \text{ and } \{(1, -2, 6) + t((1, -2, 2) \mid t \in \mathbb{R}\} \\ z &= 6 + 2t \end{aligned}$$

is the set of points in $\mathbb{R}^3$ that lie on the "enemy" line.

The "friendly" line we want is parallel to the "enemy" line and consists of the points

$$\{\, |2, 0, 1\rangle + t\,|1, -2, 2\rangle \mid t \in \mathbb{R}\}.$$

Since $|2, 0, 1\rangle + (-2) \cdot |1, -2, 2\rangle = |0, 4, -3\rangle$ then $|0, 4, -3\rangle$ is on the "friendly' line.

Example E11. Find the vector equation for the plane in $\mathbb{R}^3$ containing the points $P = |1,0,2\rangle$ and $Q = |1,2,3\rangle$ and $R = |4,5,6\rangle$.
The point $|1,0,2\rangle$ is in the plane and two vectors in the plane are

$$Q - P = |0,2,1\rangle \quad \text{and} \quad R - P = |3,5,4\rangle.$$

So the points in the plane are the points $|x,y,z\rangle$ in $\mathbb{R}^3$ which satisfy

$$|x,y,z\rangle = |1,0,2\rangle + s|0,2,1\rangle + t|3,5,4\rangle \quad \text{with } s,t \in \mathbb{R}.$$

Example E12. Where does the line

$$\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$$

intersect the plane $3x + 2y + z = 20$?

The line in parametric form is

$$\begin{aligned} x &= 1 + t, \\ y &= 2 + 2t, \qquad \text{with } t \in \mathbb{R}, \\ z &= 3 + 3t, \end{aligned}$$

and plugging into the equation of the plane gives

$$20 = 3(t+1) + 2(2t+2) + (3t+3) = 10t + 10 \text{ so that } t = 1.$$

Thus the point $|x, y, z\rangle$ with $x = 1 + 1 = 2$, $y = 2 + 2 = 4$ and $z = 3 + 3$ is on both the line and the plane.

Example E13. Find a vector form for the line of intersection of the two planes $x + 3y + 2z = 6$ and $3x + 2y + z = 11$.

The points on the intersection of the two planes are the points $|x, y, z\rangle$ that satisfy the system of equations

$$\begin{aligned} 3x + 2y - z &= 11, \\ x + 3y + 2z &= 6, \end{aligned} \qquad \text{which is} \qquad \begin{pmatrix} 3 & 2 & -1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 6 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 3 & 2 \\ 0 & -7 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -7 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

So

$$\begin{aligned} x - z &= 3, \\ y + z &= 1, \end{aligned} \qquad \text{giving} \qquad \begin{aligned} x &= 3 + z, \\ y &= 1 - z, \\ z &= 0 + z, \end{aligned}$$

where z can be any element of $\mathbb{R}$. So the solutions to these equations are

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \left\{ t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

which is the line

$$\{|3,1,0\rangle + t|1,-1,1\rangle \mid t \in \mathbb{R}\}$$

as the line of intersection of the two planes.