MAST10007 Linear Algebra

THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

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Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Lecture 8: The Hilbert space $\mathbb{R}^n$

Definition (The vector space $\mathbb{R}^n$)

Let $n \in \mathbb{Z}_{>0}$. The *$\mathbb{R}$-vector space $\mathbb{R}^n$* is

$$\mathbb{R}^n = M_{n\times 1}(\mathbb{R}) = \{|x_1, \dots, x_n\rangle \mid x_i \in \mathbb{R}\} \quad \text{where} \quad |x_1, \dots, x_n\rangle = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

The *addition and scalar multiplication* are given by

$$|x_1, x_2, \dots, x_n\rangle + |y_1, y_2, \dots, y_n\rangle = |x_1 + y_1, x_2 + y_2, \dots, x_n + y_n\rangle$$

and

$$c|x_1, x_2, \dots, x_n\rangle = |cx_1, cx_2, \dots, cx_n\rangle \quad \text{for } c \in \mathbb{R}.$$

Definition (Inner product, length function and distance function)

The *standard inner product on $\mathbb{R}^n$* is $\langle \mid \rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\langle x_1, \ldots, x_n | y_1, \ldots, y_n \rangle = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = x_1 y_1 + \cdots + x_n y_n.$$

The *length function* is $\| \; \| \colon \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ given by

$$\| \, |x_1, \ldots, x_n\rangle \| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}.$$

The *distance function* is $d \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ given by

$$d(|x_1, \ldots, x_n\rangle, |y_1, \ldots, y_n\rangle) = \| \, |x_1, \ldots, x_n\rangle - |y_1, \ldots, y_n\rangle \|.$$

Theorem (Cauchy-Schwarz and the triangle inequality)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leqslant \|\mathbf{u}\| \cdot \|\mathbf{v}\| \quad \textit{and} \quad \|\mathbf{u} + \mathbf{v}\| \leqslant \|\mathbf{u}\| + \|\mathbf{v}\|$$

If

$$\mathbf{x} = |x_1, x_2, \ldots, x_n\rangle \quad \text{and} \quad \mathbf{y} = |y_1, y_2, \ldots, y_n\rangle$$

then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

and

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} \quad \text{and} \quad \|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle \qquad \text{and}$$

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &= \|\mathbf{y} - \mathbf{x}\| = \sqrt{\langle \mathbf{y} - \mathbf{x}, \mathbf{y} - \mathbf{x} \rangle} \\ &= \sqrt{(y_1 - x_1)^2 + \cdots + (y_n - x_n)^2}. \end{aligned}$$

Properties used VERY often.

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and let $c \in \mathbb{R}$.

$$\begin{aligned}\langle \mathbf{y}, \mathbf{x} \rangle &= y_1x_1 + y_2x_2 + \cdots + y_nx_n \\ &= x_1y_1 + x_2y_2 + \cdots x_ny_n = \langle \mathbf{x}, \mathbf{y} \rangle,\end{aligned}$$

$$\langle \mathbf{x}, \mathbf{y}+\mathbf{z} \rangle = \mathbf{x}^T(\mathbf{y}+\mathbf{z}) = \mathbf{x}^T\mathbf{y} + \mathbf{x}^T\mathbf{z} = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle,$$

$$\langle \mathbf{x}+\mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{z}, \mathbf{x}+\mathbf{y} \rangle = \langle \mathbf{z}, \mathbf{x} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle,$$

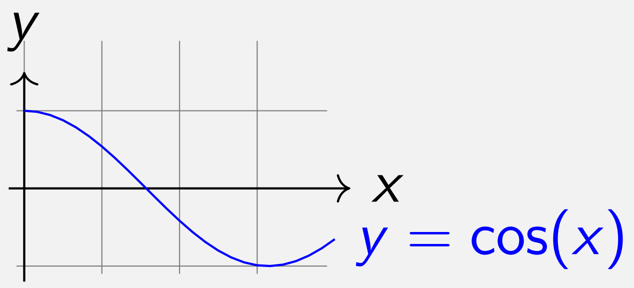
$$\langle \mathbf{x}, c\mathbf{y} \rangle = \mathbf{x}^T c\mathbf{y} = c\mathbf{x}^T\mathbf{y} = c\langle \mathbf{x}, \mathbf{y} \rangle, \langle c\mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, c\mathbf{x} \rangle = c\langle \mathbf{y}, \mathbf{x} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle,$$

$$\|c\mathbf{x}\| = \sqrt{\langle c\mathbf{x}, c\mathbf{x} \rangle} = \sqrt{c^2\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{c^2}\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = |c| \cdot \|\mathbf{x}\|.$$

Definition (Angle and projection)

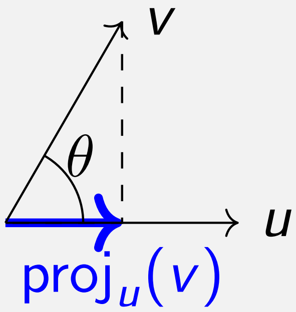
Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ with $\mathbf{u} \neq 0$ and $\mathbf{v} \neq 0$. The *angle between* $\mathbf{u}$ *and* $\mathbf{v}$ is $\theta(\mathbf{u}, \mathbf{v})$ given by

$$\cos(\theta(\mathbf{u}, \mathbf{v})) = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}.$$



The *projection of* $\mathbf{v}$ *onto* $\mathbf{u}$ is

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}.$$



Example E1. If $\mathbf{u} = |1, 3, 1, 2\rangle$ and $\mathbf{v} = |2, 1, -1, 3\rangle$ in $\mathbb{R}^4$ then

$$\mathbf{u} - \mathbf{v} = |1, -2, 0, 1\rangle$$

and the distance between the points $(1, 3, 1, 2)$ and $(2, 1, -1, 3)$ is

$$d(\mathbf{u}, \mathbf{v}) = \|\ |1, -2, 0, 1\rangle\ \| = \sqrt{1^2 + (-2)^2 + 0^2 + 1^1}$$
$$= \sqrt{1 + 4 + 0 + 1} = \sqrt{6}.$$

Example E2. If $\mathbf{u} = |0, 2, 2, -1\rangle$ and $\mathbf{v} = |-1, 1, 1, -1\rangle$ in R^4 then

$$\begin{aligned}\langle \mathbf{u}, \mathbf{v}\rangle &= \langle 0, 2, 2, -1|-1, 1, 1, -1\rangle \\ &= 0 \cdot (-1) + 2 \cdot 1 + 2 \cdot 1 + (-1) \cdot (-1) \\ &= 0 + 2 + 1 + 1 = 5\end{aligned}$$

and

$$\|\mathbf{u}\| = \sqrt{0 + 4 + 4 + 1} = \sqrt{9} = 3$$

and

$$\|\mathbf{v}\| = \sqrt{1 + 1 + 1 + 1} = \sqrt{4} = 2.$$

Since $|5| \leqslant 3 \cdot 2$ we observe that, in this case,

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leqslant \|\mathbf{u}\| \cdot \|\mathbf{v}\|.$$

Example E4. Let $\mathbf{u} = (2, -1, -2)$ and $\mathbf{v} = (2, 1, 3)$. Find vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ such that

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

where $\mathbf{v}_1$ is parallel to $\mathbf{u}$ and $\mathbf{v}_2$ is perpendicular to $\mathbf{u}$.

Solution: Since the projection of $\mathbf{v}$ onto $\mathbf{u}$ is parallel to $\mathbf{u}$ then let

$$\begin{aligned} \mathbf{v}_1 = \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u} = \frac{-3}{9}\mathbf{u} \\ = \tfrac{-1}{3}|2, -1, -2\rangle = |\tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\rangle \end{aligned}$$

and

$$\mathbf{v}_2 = \mathbf{u} - \mathbf{v}_1 = |2, -1, -2\rangle - |\tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{2}{3}\rangle = |\tfrac{8}{3}, \tfrac{2}{3}, \tfrac{7}{3}\rangle.$$

Then $\mathbf{u} = \mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1$ is parallel to $\mathbf{u}$ and $\mathbf{v}_2$ is perpendicular to $\mathbf{u}$.