MAST10007 Linear Algebra

THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

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Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Let $n \in \mathbb{Z}_{>0}$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i, j) entry and 0 elsewhere. For $i \in \{1, \ldots, n-1\}$ and $c \in \mathbb{Q}$ define

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

For $i \in \{1, \ldots, n\}$ and $d \in \mathbb{Q}$ with $d \neq 0$ define

$$h_i(d) = 1 + (d-1)E_{ii},$$

For $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and $c \in \mathbb{Q}$ define

$$x_{ij}(c) = 1 + cE_{ij},$$

For $r \in \{1, \ldots, n\}$ define

$$1_r = E_{11} + \cdots + E_{rr}.$$

Definition (Determinant)

The *deteminant* is the function $\det\colon M_{n\times n}(\mathbb{Q}) \to \mathbb{Q}$ determined by

$$\text{if } A, B \in M_{n\times n}(\mathbb{Q}) \text{ then} \quad \det(AB) = \det(A)\det(B),$$

and if $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and $r \in \{1, \ldots, n-1\}$ and $c, d \in \mathbb{Q}$ with $d \neq 0$ then

$$\det(x_{ij}(c)) = 1, \quad \det(h_i(d)) = d, \quad \det(s_r(c)) = -1.$$

Theorem (Greedy normal form)

For $r \in \{1, \ldots, \min(s, t)\}$ let

$$1_r = E_{11} + \cdots + E_{rr}.$$

Let $A \in M_{t\times s}(\mathbb{Q})$. The greedy normal form gives

$$A = (\textit{product of } s_i(c)\textit{s}) \cdot (\textit{product of } h_i(d)\textit{s}) \cdot (\textit{product of } x_{ij}(c)\textit{s}) \\ \cdot 1_r \cdot (\textit{product of } s_i(c)\textit{s}) \cdot (\textit{product of } x_{ij}(c)\textit{s}).$$

Example M12. Let $A = \begin{pmatrix} 2 & 6 & 9 \\ 0 & 3 & 8 \\ 0 & 0 & -1 \end{pmatrix}$. Then

$$
\begin{aligned}
A &= h_1(2)h_2(3)h_3(-1) \begin{pmatrix} 1 & 3 & \frac{9}{2} \\ 0 & 1 & \frac{8}{3} \\ 0 & 0 & 1 \end{pmatrix} \\
&= h_1(2)h_2(3)h_3(-1)x_{12}(3)x_{13}(\tfrac{9}{2})x_{23}(\tfrac{8}{3}),
\end{aligned}
$$

So

$$
\begin{aligned}
\det(A) &= \det\big(h_1(2)h_2(3)h_3(-1)x_{12}(3))x_{13}(\tfrac{9}{2})x_{23}(\tfrac{8}{3}\big) \\
&= \det(h_1(2))\det(h_2(3))\det(h_3(-1)) \\
&\qquad \cdot \det(x_{12}(3)))\det(x_{13}(\tfrac{9}{2}))\det(x_{23}(\tfrac{8}{3})) \\
&= 2 \cdot 3 \cdot (-1) \cdot 1 \cdot 1 \cdot 1 = -6.
\end{aligned}
$$

Example M13. Let $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Then

$$\begin{aligned} A &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{pmatrix} \\ &= s_1(-1)s_2(3) \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -7 \end{pmatrix}. \end{aligned}$$

So $\det(A) = (-1) \cdot (-1) \cdot (-1) \cdot 1 \cdot (-7) = 7$.

Definition (the (i,j)-cofactor)

Let $A \in M_{n\times n}(\mathbb{Q})$ and let $i,j \in \{1,\dots,n\}$.

Let $A^{(i,j)}$ be the matrix A with the ith row removed and the jth column removed.

The *(i,j)-cofactor of A* is

$$C_{ij} = (-1)^{i+j}\det(A^{(i,j)}).$$

Theorem (cofactor expansion)

Let $A \in M_{n\times n}(\mathbb{Q})$ and let $i,j \in \{1,\dots,n\}$. Then

$$\det(A) = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in}$$

cofactor expansion across the ith row

and

$$\det(A) = A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj}.$$

cofactor expansion down the jth column

Example M14. Since $1 = AA^{-1}$ then $\det(1) = \det(A)\det(A^{-1})$. So

$$1 = \det(A)\det(A^{-1}) \quad \text{and} \quad \frac{1}{\det(A)} = \det(A^{-1}).$$

Example M15 and 16. If $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ then the (2,3)-cofactor is

$$C_{23} = (-1)^{2+3} \det \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = (-1)^5(1 \cdot 1 - 0 \cdot 2) = -1.$$

Using cofactor expansion along the third row,

$$\begin{aligned} \det(A) = (-1)^{3+1} \cdot 0 \cdot \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} &+ (-1)^{3+2} \cdot 1 \cdot \det \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \\ &+ (-1)^{3+3} \cdot 3 \cdot \det \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \\ = 0 - (1 \cdot 1 - (-1) \cdot 1) + 3(1 \cdot 1 - (-1) \cdot 2) &= 0 - 2 + 9 = 7. \end{aligned}$$

Example M17. Let $A = \begin{pmatrix} 1 & -2 & 0 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -4 & 2 & 4 \end{pmatrix}$.

Using cofactor expansion down the fourth column,

$$\det(A) = (-1)^{1+4} \cdot 1 \cdot \det \begin{pmatrix} 3 & 2 & 2 \\ 1 & 0 & 1 \\ 0 & -4 & 2 \end{pmatrix} + 0 + 0$$

$$+ (-1)^{4+4} \cdot 4 \cdot \det \begin{pmatrix} 1 & -2 & 0 \\ 3 & 2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= -\left((-1)^{1+1} \cdot 3 \cdot \det \begin{pmatrix} 0 & 1 \\ -4 & 2 \end{pmatrix} + (-1)^{2+1} \cdot 1 \cdot \det \begin{pmatrix} 2 & 2 \\ -4 & 2 \end{pmatrix} + 0\right)$$

$$+ 4\left((-1)^{1+1} \cdot 1 \cdot \det \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix} + (-1)^{1+2} \cdot (-2) \cdot \det \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} + 0\right)$$

$$= -3(0+4) + (4+8) + 4\big((2-0) + 2(2-3)\big)$$

$$= -12 + 12 + 8 + 8 = 16.$$