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Kernels, Images and Solutions

$M_{t\times s}(\mathbb{Q}) = \{ t\times s \text{ matrices with entries in } \mathbb{Q}\}$.

$\mathbb{Q}^s = M_{s\times 1}(\mathbb{Q}) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_s \end{pmatrix} \;\middle|\; x_i \in \mathbb{Q} \right\}$

Let $A \in M_{t\times s}(\mathbb{Q})$.

$\ker(A) = \{ x \in \mathbb{Q}^s \mid Ax = 0\}$

$\mathrm{im}(A) = \{ Ax \mid x \in \mathbb{Q}^s\}$

Let $b \in \mathbb{Q}^t$, so $b = \begin{pmatrix} b_1 \\ \vdots \\ b_t \end{pmatrix}$. Then

$\mathrm{Sol}(Ax = b) = \{ x \in \mathbb{Q}^s \mid Ax = b\}$.

<u>Note:</u> $\mathrm{Sol}(Ax = 0) = \ker(A)$.

<u>Example L57</u> Let $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$

Then $Ax = b$, with $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is the system

$$\begin{aligned} x_1 + 0x_2 + x_3 &= -2 \\ 0x_1 + 2x_2 + 2x_3 &= 4 \\ 0x_1 + 0x_2 + 0x_3 &= -3. \end{aligned}$$

No choice of x_1, x_2, x_3 will satisfy the third equation.

So

$\mathrm{Sol}(Ax = b) = \emptyset$

$Ax=0$ is the system
$$x_1+0x_2+x_3=0,$$
$$0x_1+2x_2+2x_3=0,$$
$$0x_1+0x_2+0x_3=0,$$

which is $\begin{matrix} x_1+x_3=0 \\ x_2+x_3=0 \end{matrix}$ which is $\begin{matrix} x_1=0-x_3 \\ x_2=0-x_3 \\ x_3=0+x_3 \end{matrix}$

and x_3 can be anything. So

$$\ker(A)=\mathrm{Sol}(Ax=0)=\left\{\begin{pmatrix}-x_3\\-x_3\\x_3\end{pmatrix} \,\middle|\, x_3\in\mathbb{Q}\right\}$$

$$=\left\{x_3\begin{pmatrix}-1\\-1\\1\end{pmatrix} \,\middle|\, x_3\in\mathbb{Q}\right\}=\mathrm{span}\left\{\begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\}$$

$$\mathrm{im}(A)=\left\{\begin{pmatrix}1&0&1\\0&2&2\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \,\middle|\, x_1,x_2,x_3\in\mathbb{Q}\right\}$$

$$=\left\{\begin{pmatrix}x_1+0x_2+x_3\\0x_1+2x_2+2x_3\\0x_1+0x_2+0x_3\end{pmatrix} \,\middle|\, x_1,x_2,x_3\in\mathbb{Q}\right\}$$

$$=\left\{x_1\begin{pmatrix}1\\0\\0\end{pmatrix}+x_2\begin{pmatrix}0\\2\\0\end{pmatrix}+x_3\begin{pmatrix}1\\2\\0\end{pmatrix} \,\middle|\, x_1,x_2,x_3\in\mathbb{Q}\right\}$$

$$=\mathrm{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}1\\2\\0\end{pmatrix}\right\}$$

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Example LS8 If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 4 \\ 15 \end{pmatrix}$ then

$Ax = b$ with $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is the system
$$\begin{aligned} x_1 &= -2 \\ x_2 &= 4 \\ x_3 &= 15. \end{aligned}$$

So $\mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} -2 \\ 4 \\ 15 \end{pmatrix} \right\}$

$\ker(A) = \mathrm{Sol}(Ax = 0) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ since

$Ax = 0$ is the system
$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0. \end{aligned}$$

$$\begin{aligned}
\mathrm{im}(A) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{Q} \right\} \\
&= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{Q} \right\} \\
&= \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\end{aligned}$$

Example LS9 Let $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$.

If $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ then $Ax = b$ is
$$\begin{aligned} x_1 + x_2 + 5x_5 &= 1 \\ x_3 + 6x_5 &= 2 \\ x_4 + 7x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

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which is $x_1 = 1 - x_2 - 5x_5$

$x_2 = 0 + x_2 + 0x_5$

$x_3 = 2 + 0x_2 - 6x_5$

$x_4 = 0 + 0x_2 - 7x_5$

$x_5 = 0 + 0x_2 + 0x_5$

So $$\mathrm{Sol}(Ax=b) = \left\{ \begin{pmatrix}1\\0\\2\\0\\0\end{pmatrix} + x_2\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix} + x_5\begin{pmatrix}-5\\0\\-6\\-7\\0\end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{Q} \right\}$$

$$= \begin{pmatrix}1\\0\\2\\0\\0\end{pmatrix} + \left\{ x_2\begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix} + x_5\begin{pmatrix}-5\\0\\-6\\-7\\0\end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{Q} \right\}$$

$$= \begin{pmatrix}1\\0\\2\\0\\0\end{pmatrix} + \mathrm{span}\left\{ \begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-5\\0\\-6\\-7\\0\end{pmatrix} \right\}$$

Then $$\ker(A) = \mathrm{Sol}(Ax=0) = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} + \mathrm{span}\left\{ \begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-5\\0\\-6\\-7\\0\end{pmatrix} \right\}$$

$$= \mathrm{span}\left\{ \begin{pmatrix}-1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-5\\0\\-6\\-7\\0\end{pmatrix} \right\}$$