MAST10007 Linear Algebra

THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

Summer Term, 2026

Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Lecture 6: Kernels and Images

The set of $s \times 1$ matrices with entries in $\mathbb{Q}$ is

$$\mathbb{Q}^s = M_{s\times 1}(\mathbb{Q}).$$

Definition (Kernel and image of a matrix)

Let $A \in M_{t\times s}(\mathbb{Q})$. The *kernel of A* is

$$\ker(A) = \{x \in \mathbb{Q}^s \mid Ax = 0\}$$

and the *image of A* is

$$\mathrm{im}(A) = \{Ax \mid s \in \mathbb{Q}^s\}.$$

Definition (Solutions of a linear system)

Let $A \in M_{t\times s}(\mathbb{Q})$ and let $b \in \mathbb{Q}^s$. The set of *solutions of the linear system $Ax = b$* is

$$\mathrm{Sol}(Ax = b) = \{x \in \mathbb{Q}^s \mid Ax = b\}.$$

Example LS7. If $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$ then $A\mathbf{x} = \mathbf{b}$ is the system

$$\begin{aligned} x_1 + 0x_2 + x_3 &= -2, \\ 0x_1 + 2x_2 + 2x_3 &= 4, \\ 0x_1 + 0x_2 + 0x_3 &= -3, \end{aligned} \qquad \text{which has no solutions}$$

(no choice of $x_1, x_2 x_3 \in \mathbb{Q}$ will satisfy the third equation). So

$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \emptyset, \qquad \text{and the system is inconsistent.}$$

Then $A\mathbf{x} = 0$ is the system

$$\begin{aligned} x_1 + 0x_2 + x_3 &= 0, \\ 0x_1 + 2x_2 + 2x_3 &= 0, \\ 0x_1 + 0x_2 + 0x_3 &= 0, \end{aligned} \qquad \text{which is} \qquad \begin{aligned} x_1 &= -x_3, \\ x_2 &= x_3, \\ x_3 &= x_3, \end{aligned}$$

where x_3 can be any number.

So

$$\ker(A) = \left\{ x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x_3 \in \mathbb{Q} \right\} = \mathbb{Q}\text{-span} \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

Then

$$\begin{aligned} \mathrm{im}(A) &= \left\{ \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{Q} \right\} \\ &= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{Q} \right\} \\ &= \mathbb{Q}\text{-span} \left\{ \text{columns of } \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \right\} \end{aligned}$$

Example LS8. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \\ 15 \end{pmatrix}$ then

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 2, \\ 0x_1 + 1x_2 + 0x_3 &= -4, \\ 0x_1 + 0x_2 + x_3 &= 15, \end{aligned}$$

which has exactly one solution

$$\begin{aligned} x_1 &= 2, \\ x_2 &= -4, \\ x_3 &= 15. \end{aligned}$$

$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 2 \\ -4 \\ 15 \end{pmatrix} \right\}.$$

Then $A\mathbf{x} = 0$ is the system

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 0, \\ 0x_1 + 1x_2 + 0x_3 &= 0, \\ 0x_1 + 0x_2 + x_3 &= 0, \end{aligned}$$

which has exactly one solution

$$\begin{aligned} x_1 &= 0, \\ x_2 &= 0, \\ x_3 &= 0. \end{aligned}$$

So

$$\ker(A) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Then

$$\begin{aligned} \mathrm{im}(A) &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{Q} \right\} \\ &= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{Q} \right\} \\ &= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{Q} \right\} = \mathbb{Q}^3. \end{aligned}$$

Example LS9. If

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{then}$$

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + 0x_4 + 5x_5 &= 1, \\ 0x_1 + 0x_2 + 1x_3 + 0x_4 + 6x_5 &= 2, \\ 0x_1 + 0x_2 + 0x_3 + x_4 + 7x_5 &= 3, \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0, \end{aligned}$$

which has an infinite number of solutions.

More specifically,

$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \;\middle|\; \begin{array}{l} x_1 = 1 - 2x_2 - 5x_5, \\ x_2 \in \mathbb{Q}, \\ x_3 = 2 - 6x_5, \\ x_4 = 3 - 7x_5, \\ x_5 \in \mathbb{Q} \end{array} \right\}$$

Equivalently,

$$\mathrm{Sol}(A\mathbf{x}=\mathbf{b}) = \left\{ \mathbf{x} = \begin{pmatrix}1\\0\\2\\3\\0\end{pmatrix} + \begin{pmatrix}-2x_2\\x_2\\0\\0\end{pmatrix} + \begin{pmatrix}-5x_5\\0\\-6x_5\\-7x_5\\x_5\end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{Q} \right\}$$

$$= \begin{pmatrix}1\\0\\2\\3\\0\end{pmatrix} + \left\{ x_2 \begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix} + x_5 \begin{pmatrix}-5\\0\\-6\\-7\\1\end{pmatrix} \;\middle|\; x_2, x_5 \in \mathbb{Q} \right\}$$

$$= \begin{pmatrix}1\\0\\2\\3\\0\end{pmatrix} + \mathbb{Q}\text{-span} \left\{ \begin{pmatrix}-2\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}-5\\0\\-6\\-7\\1\end{pmatrix} \right\}$$

$$= \mathbf{p} + \ker(A).$$

Let $n \in \mathbb{Z}_{>0}$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i, j) entry and 0 elsewhere.

Definition (root matrices, diagonal generators and row reducers)

Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Let $c \in \mathbb{Q}$. The *root matrix $x_{ij}(c)$* is

$$x_{ij}(c) \in M_{n\times n}(\mathbb{Q}) \quad \text{given by} \quad x_{ij}(c) = 1 + cE_{ij}.$$

Let $i \in \{1, \ldots, n\}$. Let $d \in \mathbb{Q}$ with $d \neq 0$. The *diagonal generator $h_i(d)$* is

$$h_i(d) = 1 + (d-1)E_{ii}.$$

Let $i \in \{1, \ldots, n-1\}$ and let $c \in \mathbb{Q}$. The *row reducer $s_i(c)$* is

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

Theorem (Generators for GL_n)

Let $A \in GL_n(\mathbb{Q})$. Then A can be written as a product of row reducers, diagonal generators and root matrices.

The last theorem is really a special case of the following theorem.

Theorem (Greedy normal form)

For $r \in \{1, \dots, \min(s,t)\}$ let

$$1_r = E_{11} + \cdots + E_{rr}.$$

Let $A \in M_{t\times s}(\mathbb{Q})$. The greedy normal form gives

$$A = (\textit{product of } s_i(c)s) \cdot (\textit{product of } h_i(d)s) \cdot (\textit{product of } x_{ij}(c)s) \cdot 1_r \cdot (\textit{product of } s_i(c)s) \cdot (\textit{product of } x_{ij}(c)s).$$

Corollary (Packaged normal form)

Let $A \in M_{t\times s}(\mathbb{Q})$. Then there exist $P \in GL_t(\mathbb{Q})$ and $Q \in GL_s(\mathbb{Q})$ and $r \in \{1, \dots, \min(s,t)\}$ such that

$$A = P 1_r Q, \qquad \textit{where} \quad 1_r = E_{11} + E_{22} + \cdots + E_{rr}.$$

Theorem (Computing kernels and images with normal form)

Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. *Let* $r \in \{1, \dots, \min(s,t)\}$ *and let*

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \textit{in} \quad M_{t\times s}(\mathbb{Q}).$$

Then

$$\ker(P1_rQ) = \ker(1_rQ) = Q^{-1}\ker(1_r) = \textit{span}\{\textit{last } s{-}r \textit{ colums of } Q^{-1}\},$$

$$\textit{im}(P1_rQ) = \textit{im}(P1_r) = P\,\textit{im}(1_r) = \textit{span}\{\textit{first } r \textit{ colums of } P\}.$$

Corollary (rank-nullity theorem)

Let $A \in M_{t\times s}(\mathbb{Q})$. *Since* $(s-r)+r = s$ *then*

$$\dim(\ker(A)) + \dim(\textit{im}(A)) = (\textit{number of columns of } A).$$

Theorem (Computing solutions of linear systems)

Let $P \in GL_t(\mathbb{Q})$, $Q \in GL_s(\mathbb{Q})$. *Let* $r \in \{1, \ldots, \min(s,t)\}$ *and let*

$$1_r = E_{11} + E_{22} + \cdots + E_{rr} \quad \text{in} \quad M_{t \times s}(\mathbb{Q}).$$

Let $A = P 1_r Q$ *and let* $b \in \mathbb{Q}^t$. *Then*

$$Sol(Ax = b) = \begin{cases} Q^{-1} \begin{pmatrix} (P^{-1}b)_1 \\ \vdots \\ (P^{-1}b)_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ker(A), & \begin{array}{l} \text{if entries} \\ r+1, \ldots, t \\ \text{of } P^{-1}b \\ \text{are all } 0, \end{array} \\ \emptyset, & \text{otherwise.} \end{cases}$$