MAST10007 Linear Algebra

THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

Summer Term, 2026

Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Example M6 Find the inverse of $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

Start with $AA^{-1} = 1$ which is

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} -4 & -5 & -3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{pmatrix}$$

Carefully compare this solution of Example 6 in Topic 2 of the lecture slides to the solution given below.

Example 6.

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{bmatrix}$.

Solution:

Form the augmented matrix $[A \mid I_3]$ and perform row operations so that A is in reduced row-echelon form.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to R_2 + R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - 2R_3 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 3 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right] \quad R_1 \to R_1 - 2R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & -5 & 3 \\ 0 & 1 & 0 & 3 & 3 & -2 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$$

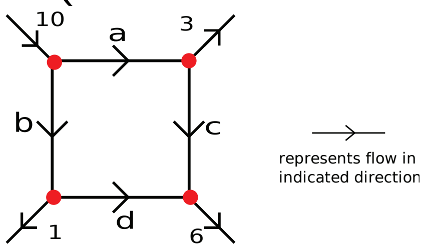
Hence

$$A^{-1} = \begin{bmatrix} -4 & -5 & 3 \\ 3 & 3 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

Example LS10. Calculating flows in networks

At each node • require (sum of flows in) = (sum of flows out).

Determine a, b, c and d in the network



Then

$$\begin{aligned} \text{Node 1:} \quad 10 &= a + b, \\ \text{Node 2:} \quad a &= 3 + c, \\ \text{Node 3:} \quad c + d &= 6, \\ \text{Node 4:} \quad b &= 1 + d. \end{aligned}$$

So

$$\begin{aligned} a + b + 0c + 0d &= 10, \\ a + 0b - c + 0d &= 2, \\ 0a + 0b + c + d &= 6, \\ 0a + b + 0c - d &= 1 \end{aligned} \quad \text{which is} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix}.$$

Start with

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 10 \\ 3 \\ 6 \\ 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 6 \\ 1 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{to get} \quad \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 1 \\ 6 \end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}3\\ 1\\ 6\\ 6\end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & -1\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}3\\ 1\\ 6\\ 0\end{pmatrix}.$$

Left multiply both sides by

$$\begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \quad \text{to get} \quad \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix}9\\ 1\\ 6\\ 0\end{pmatrix}.$$

This is the system

$$\begin{aligned} a &= 9, \\ b - d &= 1, \\ c + d &= 6, \\ 0 &= 0, \end{aligned} \qquad \text{which is} \qquad \begin{aligned} a &= 9 + 0d, \\ b &= 1 + 1d, \\ c &= 6 + (-1)d, \\ d &= 0 + 1d. \end{aligned}$$

where d can be any number. So

$$\begin{aligned} \mathrm{Sol}(A\mathbf{x} = \mathbf{b}) &= \left\{ \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + d \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; d \in \mathbb{Q} \right\} \\ &= \begin{pmatrix} 9 \\ 1 \\ 6 \\ 0 \end{pmatrix} + \mathbb{Q}\text{-span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

Definition (Transpose of a matrix)

Let $A \in M_{t\times s}(\mathbb{Q})$. The *transpose of A* is $A^T \in M_{s\times t}(\mathbb{Q})$ given by

$$(A^T)_{ij} = A_{ji}, \quad \text{for } i \in \{1, \dots, s\} \text{ and } j \in \{1, \dots, t\}.$$

Example M4. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ then $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Definition (Trace)

Let $A \in M_{n\times n}(\mathbb{Q})$. The *trace of A* is

$$\mathrm{Tr}(A) = A_{11} + \cdots + A_{nn} \quad \text{where} \quad A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

Example M3.

$$\mathrm{Tr}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 1 + 5 + 9 = 15.$$

Let $n \in \mathbb{Z}_{>0}$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i, j) entry and 0 elsewhere.

Definition (root matrices, diagonal generators and row reducers)

Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Let $c \in \mathbb{Q}$. The *root matrix $x_{ij}(c)$* is

$$x_{ij}(c) \in M_{n\times n}(\mathbb{Q}) \quad \text{given by} \quad x_{ij}(c) = 1 + cE_{ij}.$$

Let $i \in \{1, \ldots, n\}$. Let $d \in \mathbb{Q}$ with $d \neq 0$. The *diagonal generator $h_i(d)$* is

$$h_i(d) = 1 + (d-1)E_{ii}.$$

Let $i \in \{1, \ldots, n-1\}$ and let $c \in \mathbb{Q}$. The *row reducer $s_i(c)$* is

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

Row operations

Let

$$A = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \quad \text{and} \quad x_{13}(54) = \begin{pmatrix} 1 & 0 & 54 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Left multiplying by $x_{13}(54)$ adds $54 \cdot (\text{row } 3)$ to row 1:

$$\begin{aligned} x_{13}(54)A &= \begin{pmatrix} 1 & 0 & 54 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \\ &= \begin{pmatrix} 16203 & -5409 & 10807 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix}. \end{aligned}$$

Row operations

Let

$$A = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \quad \text{and} \quad h_3(6) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Left multiplying by $h_3(6)$ multiplies row 3 by 6:

$$h_3(6)A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 1800 & -600 & 1200 \end{pmatrix}.$$

Row operations

Let

$$A = \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \quad \text{and} \quad s_2(-5) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Left multiplying by $s_2(-5)$ moves row 2 to be row 3 and makes row 2 equal to $(-5) \cdot (\text{row } 2) + (\text{row } 3)$:

$$\begin{aligned} s_2(-5) \cdot A &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -5 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -9 & 7 \\ 13 & -21 & 35 \\ 300 & -100 & 200 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -9 & 7 \\ 235 & 5 & 25 \\ 13 & -21 & 35 \end{pmatrix}. \end{aligned}$$

What are the $s_i(c)$ matrices?

In math: Let $n \in \mathbb{Z}_{>0}$ and let E_{ij} be the $n \times n$ matrix with 1 in the (i, j) entry and 0 elsewhere. For $i \in \{1, \ldots, n-1\}$ and $p, q \in \mathbb{Z}$ with $q \neq 0$ define

$$s_i\left(\tfrac{p}{q}\right) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} - E_{i+1,i} + \tfrac{p}{q} E_{i,i+1}.$$

In English: $s_i\left(\frac{p}{q}\right)$ is the $n \times n$ matrix with

(a) 1s on the diagonal except that the (i, i) enrty is c and the $(i+1, i+1)$ entry is 0, and

(b) all other entries are 0 except that the $(i, i+1)$ entry is 1 and the $(i+1, i)$ entry is 0.

What are the $s_i(c)$ matrices?

By Cartoon: If $n = 8$ and $\frac{p}{q} = \frac{7}{12}$ then

$$s_6\left(\tfrac{7}{12}\right) = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & \frac{7}{12} & 1 & \\ & & & & & 1 & 0 & \\ & & & & & & & 1 \end{pmatrix}$$

Note

$$s_6\left(\tfrac{7}{12}\right)^{-1} = \begin{pmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & 1 & & & & \\ & & & & 1 & & & \\ & & & & & 0 & 1 & \\ & & & & & 1 & -\frac{7}{12} & \\ & & & & & & & 1 \end{pmatrix}$$

We will use the $s_i(\frac{p}{q})$ (sometimes the $s_i(\frac{p}{q})^{-1}$ matrices to,

step by step, make lower triangular entries 0.

Make lower triangular entries 0 in this order

$$\begin{pmatrix} * & * & * & * & * \\ 4 & * & * & * & * \\ 3 & 7 & * & * & * \\ 2 & 6 & 9 & * & * \\ 1 & 5 & 8 & 10 & * \end{pmatrix}$$

To make the (nonzero) (i, j) entry of the matrix A into 0

In Math: Let q be the (i, j)-entry of A and let p be the $(i-1, j)$ entry of A. Assume $q \neq 0$. Then

$$A = s_{i-1}\left(\tfrac{p}{q}\right)B, \qquad \text{where} \quad B = s_{i-1}\left(\tfrac{p}{q}\right)^{-1}A.$$

In Cartoon: Suppose

$$A = \begin{matrix} \\ \\ i-1 \\ i \\ \\ \end{matrix} \begin{pmatrix} & & & \text{STUFF} & & & \\ & & & & & & \\ 0 & 0 & p & r & t & v & x \\ 0 & 0 & q & s & u & w & y \\ 0 & 0 & 0 & z & e & f & g \\ & & & & & & \end{pmatrix}, \quad \text{with } q \neq 0.$$

In Cartoon: Suppose

$$A = \begin{matrix} \\ \\ i-1 \\ i \\ \\ \\ \end{matrix} \left(\begin{array}{ccccccc} & & & \text{STUFF} & & & \\ & & & & & & \\ 0 & 0 & p & r & t & v & x \\ 0 & 0 & q & s & u & w & y \\ 0 & 0 & 0 & z & e & f & g \\ & & & & & & \end{array} \right), \quad \text{with } q \neq 0.$$

Then

$$A = s_{i-1}\left(\frac{p}{q}\right) B,$$

where

$$B = \begin{matrix} \\ \\ i-1 \\ i \\ \\ \\ \end{matrix} \left(\begin{array}{ccccccc} & & & \text{STUFF} & & & \\ & & & & & & \\ 0 & 0 & q & s & u & w & y \\ 0 & 0 & 0 & r - \frac{p}{q}s & t - \frac{p}{q}u & v - \frac{p}{q}w & x - \frac{p}{q}y \\ 0 & 0 & 0 & z & e & f & g \\ & & & & & & \end{array} \right).$$

In English: Let q be the (i,j) entry of A. If $q \neq 0$ then make the (i,j) into 0 as follows. Let p be the $(i-1,j)$ entry of A Then write

$$A = s_{i-1}\left(\tfrac{p}{q}\right)B, \qquad \text{where}$$

(a) The ith row of A moves up one row to become the $(i-1)$st row of B,

(b) The ith row of B is ((the $(i-1)$st row of A)-c(ith row of A)), and

(c) all other rows of B are the same as the corresponding rows fo A.

In hybrid:

$$A = s_{i-1}\left(\tfrac{p}{q}\right)B, \qquad \text{where}$$

(a) $\mathrm{row}_{i-1}(B) = \mathrm{row}_i(A)$,

(b) $\mathrm{row}_i(B) = \mathrm{row}_{i-1}(A) - \frac{p}{q}\mathrm{row}_i(A)$,

(c) if $j \notin \{i-1, i\}$ then $\mathrm{row}_j(B) = \mathrm{row}_(A)$,