An application of matrices: Solving linear systems

If $A = \begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

then

$Ax = b$ is the same as $\begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

is the same as $\begin{pmatrix} 3x_1 + x_2 \\ -x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

is the same as

$$3x_1 + x_2 = 7,$$
$$-x_1 + 4x_2 = 2.$$

A <u>linear system</u> is the same as $Ax = b$.

The <u>set of solutions of $Ax = b$</u> is

$$Sol(Ax = b) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; Ax = b \right\}$$

The linear system $Ax = b$ is <u>inconsistent</u> if

$$Sol(Ax = b) = \emptyset$$

The linear system $Ax = b$ is <u>consistent</u> if

$$Sol(Ax = b) \neq \emptyset.$$

<u>Example</u> If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then $Ax = b$ is the linear system

$$\begin{matrix} x_1 + 0x_2 = 7 \\ 0x_1 + 0x_2 = 2 \end{matrix} \quad \text{or} \quad \begin{matrix} x_1 = 7 \\ 0 = 2. \end{matrix}$$

So there are <u>NO</u> solutions.

$$\mathrm{Sol}(Ax = b) = \emptyset$$

The system is inconsistent

<u>Example</u> If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then $Ax = b$ with $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the linear system

$$\begin{matrix} x_1 + 0x_2 = 7 \\ 0x_1 + x_2 = 2 \end{matrix} \quad \text{or} \quad \begin{matrix} x_1 = 7 \\ x_2 = 2. \end{matrix}$$

So there is exactly <u>one</u> solution

$$\mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} 7 \\ 2 \end{pmatrix} \right\}$$

The system is consistent

<u>Example</u> If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $b = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ then $Ax = b$ is the linear system

$$\begin{matrix} x_1 + 0x_2 = 7 \\ 0x_1 + 0x_2 = 0 \end{matrix} \quad \text{or} \quad \begin{matrix} x_1 = 7 \\ x_2 = \text{anything.} \end{matrix}$$

LOTS of solutions.

$$\mathrm{Sol}(Ax=b) = \left\{ \begin{pmatrix} 7 \\ c \end{pmatrix} \;\middle|\; c \in \mathbb{Q} \right\}$$

$$= \left\{ \begin{pmatrix} 7 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{Q} \right\}$$

$$= \begin{pmatrix} 7 \\ 0 \end{pmatrix} + \left\{ c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{Q} \right\}$$

$$= \begin{pmatrix} 7 \\ 0 \end{pmatrix} + \mathbb{Q}\text{-span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

<u>Example LS2</u> If $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

then $Ax = b$ is
$$2x - y = 3.$$
$$x + y = 0.$$

Start with $Ax = b$,

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

Left multiply both sides by $\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Left multiply by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{or} \quad \begin{array}{l} x = 1 \\ y = -1. \end{array}$$

So $\mathrm{Sol}(Ax = b) = \left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$.

<u>Example LS5</u> Solve

$$\begin{aligned} 4x - 2y + 5z &= 31, \\ 2x - 3y - 2z &= 13, \\ x - 3y + 2z &= 11. \end{aligned}$$

This $A\vec{x} = b$ with

$$A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad b = \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix}.$$

Start with

$$\begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix}$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ to get

$$\begin{pmatrix} 4 & -2 & 5 \\ 1 & -3 & 2 \\ 0 & 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31 \\ 11 \\ -9 \end{pmatrix}$$

Left multiply both sides by $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -3 \\ 0 & 3 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -13 \\ -9 \end{pmatrix}$$

Left multiply both sides by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{10}{3} \end{pmatrix}$$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 17 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \\ 17 \end{pmatrix}$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{17} \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \\ 1 \end{pmatrix}$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \\ 1 \end{pmatrix}$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ 1 \end{pmatrix}$$

Left multiply both sides by $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{matrix} x = 6 \\ y = -1 \\ z = 1 \end{matrix}$$

So

$$\mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \right\}.$$