Lecture 4: Solutions of linear systems

If

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

then

$$A\mathbf{x} = \mathbf{b} \quad \text{is the same as} \quad \begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

is the same as

$$\begin{pmatrix} 3x_1 + x_2 \\ -x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

which is the same as

$$\begin{aligned} 3x_1 + x_2 &= 7 \\ -x_1 + 4x_2 &= 2 \end{aligned}$$

In general $A\mathbf{x} = \mathbf{b}$ looks like

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Definition (Solutions of a linear system)

Let $A \in M_{m\times n}(\mathbb{Q})$ and $b \in M_{n\times 1}(\mathbb{Q})$. The set of solutions of $A\mathbf{x} = \mathbf{b}$ is

$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M_{n\times 1}(\mathbb{Q}) \;\middle|\; A\mathbf{x} = \mathbf{b} \right\}.$$

The system $A\mathbf{x} = \mathbf{b}$ is *inconsistent* if $\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \emptyset$.
The system $A\mathbf{x} = \mathbf{b}$ is *consistent* if $\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) \neq \emptyset$.

Example A. If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then $\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 7 \\ 2 \end{pmatrix} \right\}$

and $\begin{aligned} x_1 + 0x_2 &= 7, \\ 0x_1 + x_2 &= 2. \end{aligned}$ has exactly one solution $\begin{aligned} x_1 &= 7, \\ x_2 &= 2. \end{aligned}$

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then $\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \emptyset$ and

$\begin{aligned} x_1 + 0x_2 &= 7, \\ x_1 + 0x_2 &= 2. \end{aligned}$ has no solutions.

If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ then $\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 7 \\ c \end{pmatrix} \;\middle|\; c \in \mathbb{Q} \right\}$,

$\begin{aligned} x_1 + 0x_2 &= 7, \\ 0x_1 + 0x_2 &= 0. \end{aligned}$ has infinitely many solutions $\begin{aligned} x_1 &= 7, \\ x_2 &= c, \end{aligned}$ for any $c \in \mathbb{Q}$.

Example LS2,3&4. If $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ then $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \text{which is} \quad \begin{aligned} 2x - y &= 3, \\ x + y &= 0. \end{aligned}$$

Start with

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{which is} \quad \begin{matrix} x = 1, \\ y = -1. \end{matrix}$$

So $\qquad \mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$ $\qquad$ (exactly *one* solution).

Example LS5&6. Solve the following system of linear equations.

$$4x - 2y + 5z = 31,$$
$$2x - 3y - 2z = 13,$$
$$x - 3y + 2z = 11.$$

In matrix form, this is $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix}.$$

Start with

$$\begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ to get

$$\begin{pmatrix} 4 & -2 & 5 \\ 1 & -3 & 2 \\ 0 & 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 31 \\ 11 \\ -9 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -3 \\ 0 & 3 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -13 \\ -9 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{10}{3} \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 17 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -9 \\ 17 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{17} \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \\ 1 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \\ 1 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \\ 1 \end{pmatrix}.$$

Left multiply both sides by $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ to get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}, \qquad \text{or} \qquad \begin{aligned} x &= 6, \\ y &= -1, \\ z &= 1. \end{aligned}$$

So

$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \right\} \qquad \text{(exactly } \textit{one} \text{ solution).}$$

Theorem

If $A \in GL_n(\mathbb{Q})$ then every linear system of the form $A\mathbf{x} = \mathbf{b}$ has a unique solution, given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

So, if $A \in GL_n\mathbb{Q})$ then

$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \{A^{-1}\mathbf{b}\}, \quad \textit{which contains exactly one element.}$$

This is because, by multiplying $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1},

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}, \qquad \text{which says} \quad \mathbf{x} = A^{-1}\mathbf{b}.$$

Let $n \in \mathbb{Z}_{>0}$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i, j) entry and 0 elsewhere.

Definition (root matrices, diagonal generators and row reducers)

Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Let $c \in \mathbb{Q}$. The *root matrix $x_{ij}(c)$* is

$$x_{ij}(c) \in M_{n\times n}(\mathbb{Q}) \quad \text{given by} \quad x_{ij}(c) = 1 + cE_{ij}.$$

Let $i \in \{1, \ldots, n\}$. Let $d \in \mathbb{Q}$ with $d \neq 0$. The *diagonal generator $h_i(d)$* is

$$h_i(d) = 1 + (d-1)E_{ii}.$$

Let $i \in \{1, \ldots, n-1\}$ and let $c \in \mathbb{Q}$. The *row reducer $s_i(c)$* is

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

Theorem (Generators for GL_n)

Let $A \in GL_n(\mathbb{Q})$. Then A can be written as a product of row reducers, diagonal generators and root matrices.

The last theorem is really a special case of the following theorem.

Theorem (Greedy normal form)

For $r \in \{1, \ldots, \min(s,t)\}$ *let*

$$1_r = E_{11} + \cdots + E_{rr}.$$

Let $A \in M_{t\times s}(\mathbb{Q})$*. The greedy normal form gives*

$$A = (\textit{product of } s_i(c)s) \cdot (\textit{product of } h_i(d)s) \cdot (\textit{product of } x_{ij}(c)s) \cdot 1_r \cdot (\textit{product of } s_i(c)s) \cdot (\textit{product of } x_{ij}(c)s).$$

Corollary (Packaged normal form)

Let $A \in M_{t\times s}(\mathbb{Q})$*. Then there exist* $P \in GL_t(\mathbb{Q})$ *and* $Q \in GL_s(\mathbb{Q})$ *and* $r \in \{1, \ldots, \min(s,t)\}$ *such that*

$$A = P 1_r Q, \qquad \textit{where} \quad 1_r = E_{11} + E_{22} + \cdots + E_{rr}.$$

Example LS5. Find the greedy normal form of $A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix}$.

$$
\begin{aligned}
A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix} &= s_2(2) \begin{pmatrix} 4 & -2 & 5 \\ 1 & -3 & 2 \\ 0 & 3 & -6 \end{pmatrix} \\
&= s_2(2)s_1(4) \begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -3 \\ 0 & 3 & -6 \end{pmatrix} = s_2(2)s_1(4)s_2(\tfrac{10}{3}) \begin{pmatrix} 1 & -3 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 17 \end{pmatrix} \\
&= s_2(2)s_1(4)s_2(\tfrac{10}{3})h_2(3)h_3(17) \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \\
&= s_2(2)s_1(4)s_2(\tfrac{10}{3})h_2(3)h_3(17)x_{23}(-2)x_{13}(2)x_{12}(-3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= s_2(2)s_1(4)s_2(\tfrac{10}{3})h_2(3)h_3(17)x_{23}(-2)x_{13}(2)x_{12}(-3).
\end{aligned}
$$