06.01.2026
Linear Algebra ①
Lect. 3 A. Ram

Solutions of linear systems

If $A = \begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

then

$Ax = b$ is the same as $\begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

is the same as $\begin{pmatrix} 3x_1 + x_2 \\ -x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

is the same as
$$\begin{aligned} 3x_1 + x_2 &= 7 \\ -x_1 + 4x_2 &= 2 \end{aligned}$$

A <u>linear system</u> is the same as $Ax = b$

The <u>set of solutions of $Ax=b$</u> is

$$\mathrm{Sol}(Ax=b) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \,\middle|\, Ax = b \right\}.$$

<u>Example</u> If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then

$Ax = b$ is the linear system
$$\begin{aligned} x_1 + 0x_2 &= 7 \\ 0x_1 + x_2 &= 2 \end{aligned}$$

which is $\begin{aligned} x_1 &= 7 \\ x_2 &= 2. \end{aligned}$ So there is <u>exactly one</u> solution.

$$\mathrm{Sol}(Ax=b) = \left\{ \begin{pmatrix} 7 \\ 2 \end{pmatrix} \right\} \quad \text{or} \quad \begin{aligned} x_1 &= 7 \\ x_2 &= 2 \end{aligned}$$

Example

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$. Then

$Ax = b$ is the linear system
$$\begin{aligned} x_1 + 0x_2 &= 7 \\ 0x_1 + 0x_2 &= 2 \end{aligned}$$

which is $\begin{aligned} x_1 &= 0 \\ 0 &= 2. \end{aligned}$ So there are NO solutions.

$$\mathrm{Sol}(Ax = b) = \emptyset.$$

Example $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$. Then

$Ax = b$ is the linear system
$$\begin{aligned} x_1 + 0x_2 &= 7 \\ 0x_1 + 0x_2 &= 0 \end{aligned}$$

which is $x_1 = 7$, $x_2 =$ anything. Infinitely many solutions.

$$\begin{aligned} \mathrm{Sol}(Ax = b) &= \left\{ \begin{pmatrix} 7 \\ c \end{pmatrix} \;\middle|\; c \in \mathbb{Q} \right\} \\ &= \left\{ \begin{pmatrix} 7 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{Q} \right\} \\ &= \begin{pmatrix} 7 \\ 0 \end{pmatrix} + \left\{ c\begin{pmatrix} 0 \\ 1 \end{pmatrix} \;\middle|\; c \in \mathbb{Q} \right\} \\ &= \begin{pmatrix} 7 \\ 0 \end{pmatrix} + \mathbb{Q}\text{-span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

<u>Example L52</u> If $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$

then $Ax = b$ is
$$\begin{aligned} 2x - y &= 3 \\ x + y &= 0. \end{aligned}$$

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = s_1(2) \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} = s_1(2) h_1(-3) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= s_1(2) h_1(-3) x_{12}(1)$$

Then multiply $A\vec{x} = b$ on the left by A^{-1} to get

$$\vec{x} = A^{-1} b = x_{12}(1)^{-1} h_1(-3)^{-1} s_1(2)^{-1} b$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{1}{3} \\ 0 & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{or} \quad \begin{aligned} x &= 1 \\ y &= -1. \end{aligned}$$

So
$$\mathrm{Sol}(Ax = b) = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$ <u>one solution</u>

Example LS 5 Solve

$4x-2y+5z=31$
$2x-3y-2z=13$
$x-3y+2z=11$

which is $A\vec{x}=b$ where

$$A=\begin{pmatrix}4&-2&5\\2&-3&-2\\1&-3&2\end{pmatrix},\quad \vec{x}=\begin{pmatrix}x\\y\\z\end{pmatrix} \text{ and } b=\begin{pmatrix}31\\13\\11\end{pmatrix}$$

Then

$$A=\begin{pmatrix}4&-2&5\\2&-3&-2\\1&-3&2\end{pmatrix}=s_2\left(\tfrac{2}{1}\right)\begin{pmatrix}4&-2&5\\1&-3&2\\0&3&-6\end{pmatrix}$$

$$=s_2(2)s_1\left(\tfrac{4}{1}\right)\begin{pmatrix}1&-3&2\\0&10&-3\\0&3&-6\end{pmatrix}$$

$$=s_2(2)s_1(4)s_2\left(\tfrac{10}{3}\right)\begin{pmatrix}1&-3&2\\0&3&-6\\0&0&17\end{pmatrix}$$

$$=s_2(2)s_1(4)s_2\left(\tfrac{10}{3}\right)\begin{pmatrix}1&0&0\\0&3&0\\0&0&17\end{pmatrix}\begin{pmatrix}1&-3&2\\0&1&-2\\0&0&1\end{pmatrix}$$

$$=s_2(2)s_1(4)s_2\left(\tfrac{10}{3}\right)h_1(1)h_2(3)h_3(17)\begin{pmatrix}1&-3&2\\0&1&-2\\0&0&1\end{pmatrix}$$

$$=s_2(2)s_1(4)s_2\left(\tfrac{10}{3}\right)h_1(1)h_2(3)h_3(17)$$
$$\cdot x_{23}(-2)x_{13}(2)x_{12}(-3).$$

Multiplying $Ax=b$ by A^{-1} gives

$$\vec{x} = A^{-1}b = x_{12}(-3)^{-1} x_{13}(2)^{-1} x_{23}(-2)^{-1} h_3(17)^{-1} h_2(3)^{-1}$$

$$\cdot s_2\left(\tfrac{10}{3}\right)^{-1} s_1(4)^{-1} s_2(2)^{-1} b$$

$$= \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{17} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -\frac{10}{3} \end{pmatrix}$$

$$\cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{17} \\ 0 & 0 & \frac{1}{17} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -4 & -\frac{10}{3} \end{pmatrix} \begin{pmatrix} 31 \\ 11 \\ -9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & \frac{4}{17} \\ 0 & \frac{1}{3} & \frac{2}{17} \\ 0 & 0 & \frac{1}{17} \end{pmatrix} \begin{pmatrix} 11 \\ -9 \\ 17 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{matrix} x = 6 \\ y = -1 \\ z = 1 \end{matrix}$$

So

$$\mathrm{Sol}(Ax=b) = \left\{ \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \right\}$$

exactly
<u>one</u> solution