MAST10007 Linear Algebra

THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

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Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Lecture 3: Solutions of linear systems

If

$$A = \begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

then

$$A\mathbf{x} = \mathbf{b} \quad \text{is the same as} \quad \begin{pmatrix} 3 & 1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

is the same as

$$\begin{pmatrix} 3x_1 + x_2 \\ -x_1 + 4x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

which is the same as

$$\begin{aligned} 3x_1 + x_2 &= 7 \\ -x_1 + 4x_2 &= 2 \end{aligned}$$

In general $A\mathbf{x} = \mathbf{b}$ looks like

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Definition (Solutions of a linear system)

Let $A \in M_{m\times n}(\mathbb{Q})$ and $b \in M_{n\times 1}(\mathbb{Q})$. The set of solutions of $A\mathbf{x} = \mathbf{b}$ is

$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M_{n\times 1}(\mathbb{Q}) \;\middle|\; A\mathbf{x} = \mathbf{b} \right\}.$$

Theorem

If $A \in GL_n(\mathbb{Q})$ then every linear system of the form $A\mathbf{x} = \mathbf{b}$ has a unique solution, given by $\mathbf{x} = A^{-1}\mathbf{b}$.

Example A. If $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then $\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 7 \\ 2 \end{pmatrix} \right\}$

and $\begin{aligned} x_1 + 0x_2 &= 7, \\ 0x_1 + x_2 &= 2. \end{aligned}$ has exactly one solution $\begin{aligned} x_1 &= 7, \\ x_2 &= 2. \end{aligned}$

If $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$ then $\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \emptyset$ and

$\begin{aligned} x_1 + 0x_2 &= 7, \\ x_1 + 0x_2 &= 2. \end{aligned}$ has no solutions.

If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$ then $\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 7 \\ c \end{pmatrix} \;\middle|\; c \in \mathbb{Q} \right\}$,

$\begin{aligned} x_1 + 0x_2 &= 7, \\ 0x_1 + 0x_2 &= 0. \end{aligned}$ has infinitely many solutions $\begin{aligned} x_1 &= 7, \\ x_2 &= c, \end{aligned}$ for any $c \in \mathbb{Q}$.

Let $n \in \mathbb{Z}_{>0}$. Let E_{ij} be the $n \times n$ matrix with 1 in the (i, j) entry and 0 elsewhere.

Definition (root matrices, diagonal generators and row reducers)

Let $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Let $c \in \mathbb{Q}$. The *root matrix $x_{ij}(c)$* is

$$x_{ij}(c) \in M_{n\times n}(\mathbb{Q}) \quad \text{given by} \quad x_{ij}(c) = 1 + cE_{ij}.$$

Let $i \in \{1, \ldots, n\}$. Let $d \in \mathbb{Q}$ with $d \neq 0$. The *diagonal generator $h_i(d)$* is

$$h_i(d) = 1 + (d - 1)E_{ii}.$$

Let $i \in \{1, \ldots, n - 1\}$ and let $c \in \mathbb{Q}$. The *row reducer $s_i(c)$* is

$$s_i(c) = 1 - E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i} + cE_{ii}.$$

Theorem (Generators for GL_n)

Let $A \in GL_n(\mathbb{Q})$. Then A can be written as a product of row reducers, diagonal generators and root matrices.

The last theorem is really a special case of the following theorem.

Theorem (Greedy normal form)

For $r \in \{1, \ldots, \min(s,t)\}$ *let*

$$1_r = E_{11} + \cdots + E_{rr}.$$

Let $A \in M_{t\times s}(\mathbb{Q})$. *The greedy normal form gives*

$$\begin{aligned} A = (\textit{product of } s_i(c)s) \cdot (\textit{product of } h_i(d)s) \cdot (\textit{product of } x_{ij}(c)s) \\ \cdot 1_r \cdot (\textit{product of } s_i(c)s) \cdot (\textit{product of } x_{ij}(c)s). \end{aligned}$$

Corollary (Packaged normal form)

Let $A \in M_{t\times s}(\mathbb{Q})$. *Then there exist* $P \in GL_t(\mathbb{Q})$ *and* $Q \in GL_s(\mathbb{Q})$ *and* $r \in \{1, \ldots, \min(s,t)\}$ *such that*

$$A = P 1_r Q, \qquad \textit{where} \quad 1_r = E_{11} + E_{22} + \cdots + E_{rr}.$$

Example LS2,3&4. If $A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ then $A\mathbf{x} = \mathbf{b}$ is

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \quad \text{which is} \quad \begin{aligned} 2x - y &= 3, \\ x + y &= 0. \end{aligned}$$

The greedy normal form of A is

$$\begin{aligned} A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} &= s_1(2) \begin{pmatrix} 1 & 1 \\ 0 & -3 \end{pmatrix} = s_1(2)h_2(-3) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= s_1(2)h_2(-3)x_{12}(1). \end{aligned}$$

Multiplying $A\mathbf{x} = \mathbf{b}$ on the left by A^{-1} gives

$$\begin{aligned} \mathbf{x} = A^{-1}\mathbf{b} &= x_{12}(-1)h_2(-\tfrac{1}{3})s_1(2)^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix} \\ &= x_{12}(-1)h_3(-\tfrac{1}{3}) \begin{pmatrix} 0 \\ 3 \end{pmatrix} = x_{12}(-1) \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned}$$

or $x = 1$, $y = 1$.

So $\quad \mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} \quad$ (exactly *one* solution).

Example LS5. Find the greedy normal form of $A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix}$.

$$
\begin{aligned}
A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix} &= s_2(2) \begin{pmatrix} 4 & -2 & 5 \\ 1 & -3 & 2 \\ 0 & 3 & -6 \end{pmatrix} \\
&= s_2(2)s_1(4) \begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -3 \\ 0 & 3 & -6 \end{pmatrix} = s_2(2)s_1(4)s_2(\tfrac{10}{3}) \begin{pmatrix} 1 & -3 & 2 \\ 0 & 3 & -6 \\ 0 & 0 & 17 \end{pmatrix} \\
&= s_2(2)s_1(4)s_2(\tfrac{10}{3})h_2(3)h_3(17) \begin{pmatrix} 1 & -3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} \\
&= s_2(2)s_1(4)s_2(\tfrac{10}{3})h_2(3)h_3(17)x_{23}(-2)x_{13}(2)x_{12}(-3) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= s_2(2)s_1(4)s_2(\tfrac{10}{3})h_2(3)h_3(17)x_{23}(-2)x_{13}(2)x_{12}(-3).
\end{aligned}
$$

Example LS5&6. Solve the following system of linear equations.

$$\begin{aligned} 4x - 2y + 5z &= 31, \\ 2x - 3y - 2z &= 13, \\ x - 3y + 2z &= 11. \end{aligned}$$

In matrix form, this is $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 4 & -2 & 5 \\ 2 & -3 & -2 \\ 1 & -3 & 2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 31 \\ 13 \\ 11 \end{pmatrix}.$$

The greedy normal form of A is

$$\begin{aligned} A &= s_2(2)s_1(4)s_2(\tfrac{10}{3})h_2(3)h_3(17)x_{23}(-2)x_{13}(2)x_{12}(-3) \quad \text{and} \\ A^{-1} &= x_{12}(-3)^{-1}x_{13}(2)^{-1}x_{23}(-2)^{-1}h_3(17)^{-1}h_2(3)^{-1} \\ &\qquad \cdot s_2(\tfrac{10}{3})^{-1}s_1(4)^{-1}s_2(2)^{-1}. \end{aligned}$$

and so $\mathbf{x} = A^{-1}\mathbf{b}$ and

$$\begin{aligned}
\mathbf{x} &= x_{12}(3)x_{13}(-2)x_{23}(2)h_3(\tfrac{1}{17})h_2(\tfrac{1}{3})s_2(\tfrac{10}{3})^{-1}s_1(4)^{-1}s_2(2)^{-1}\begin{pmatrix}31\\13\\11\end{pmatrix}\\
&= x_{12}(3)x_{13}(-2)x_{23}(2)h_3(\tfrac{1}{17})h_2(\tfrac{1}{3})s_2(\tfrac{10}{3})^{-1}s_1(4)^{-1}\begin{pmatrix}31\\11\\-9\end{pmatrix}\\
&= x_{12}(3)x_{13}(-2)x_{23}(2)h_3(\tfrac{1}{17})h_2(\tfrac{1}{3})s_2(\tfrac{10}{3})^{-1}\begin{pmatrix}11\\-13\\-9\end{pmatrix}\\
&= x_{12}(-3)x_{13}(-2)x_{23}(2)h_3(\tfrac{1}{17})h_2(\tfrac{1}{3})\begin{pmatrix}11\\-9\\17\end{pmatrix}\\
&= x_{12}(-3)x_{13}(-2)x_{23}(2)\begin{pmatrix}11\\-3\\1\end{pmatrix}.
\end{aligned}$$

So

$$\mathbf{x} = x_{12}(-3)x_{13}(-2)x_{23}(2)\begin{pmatrix} 11 \\ -3 \\ 1 \end{pmatrix}$$

$$= x_{12}(-3)x_{13}(-2)\begin{pmatrix} 11 \\ -1 \\ 1 \end{pmatrix} = x_{12}(-3)\begin{pmatrix} 9 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$$

and

$$\mathrm{Sol}(A\mathbf{x} = \mathbf{b}) = \left\{ \begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix} \right\} \qquad \text{(exactly } \textit{one} \text{ solution).}$$