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Linear Algebra
Lect. 2
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The set of <u>invertible n×n matrices</u>

is $GL_n(\mathbb{Q}) = \left\{ A \in M_{n\times n}(\mathbb{Q}) \;\middle|\; \begin{array}{l}\text{there exists}\\ A^{-1} \text{ in } M_{n\times n}(\mathbb{Q}) \text{ with}\\ AA^{-1}=1 \text{ and } A^{-1}A=1\end{array} \right\}$

<u>Inverses of products</u>

Suppose $A, B, C, D \in GL_n(\mathbb{Q})$ and

$$M = ABCD.$$

Then $M^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}$ since

$$\begin{aligned} MD^{-1}C^{-1}B^{-1}A^{-1} &= ABCDD^{-1}C^{-1}B^{-1}A^{-1} \\ &= ABCC^{-1}B^{-1}A^{-1} \\ &= ABB^{-1}A^{-1} = AA^{-1} = 1 \end{aligned}$$

and

$$\begin{aligned} D^{-1}C^{-1}B^{-1}A^{-1}M &= D^{-1}C^{-1}B^{-1}A^{-1}ABCD \\ &= D^{-1}C^{-1}B^{-1}BCD \\ &= D^{-1}C^{-1}CD = D^{-1}D = 1. \end{aligned}$$

So, if you know $A^{-1}, B^{-1}, C^{-1}, D^{-1}$ then finding M^{-1} is easy.

If

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

then

$$s_1(c)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -c & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c \end{pmatrix}$$

If

$$h_1(d) = \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_2(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_3(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

then

$$h_1(d)^{-1} = \begin{pmatrix} d^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_2(d)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad h_3(d)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d^{-1} \end{pmatrix}$$

If

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$x_{12}(c)^{-1} = \begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{13}(c)^{-1} = \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{23}(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}.$$

Example

$$\begin{pmatrix} 1 & 10 & 9 & 7 & 4 \\ 0 & 1 & 8 & 6 & 3 \\ 0 & 0 & 1 & 5 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = x_{45}(1)\, x_{35}(2)\, x_{25}(3)\, x_{15}(4) \cdot x_{34}(5)\, x_{24}(6)\, x_{14}(7) \cdot x_{23}(8)\, x_{13}(9) \cdot x_{12}(10).$$

$$= x_{45}(1)\, x_{35}(2)\, x_{25}(3)\, x_{15}(4)\, x_{34}(5)\, x_{24}(6)\, x_{14}(7)\, x_{23}(8)\, x_{13}(9)\, x_{12}(10).$$

The order is important!

<u>Example M8</u> Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Call the oracle to find out

$$A = s_1\left(\tfrac{1}{3}\right) h_1(3)\, h_2\left(\tfrac{2}{3}\right) x_{12}\left(\tfrac{4}{3}\right), \quad \text{where } x_{12}\left(\tfrac{4}{3}\right) = \begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{pmatrix}$$

$$s_1\left(\tfrac{1}{3}\right) = \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 0 \end{pmatrix}, \quad h_1(3) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \quad h_2\left(\tfrac{2}{3}\right) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}.$$

If you don't trust the oracle, check:

$$s_1\left(\tfrac{1}{3}\right) h_1(3)\, h_2\left(\tfrac{2}{3}\right) x_{12}\left(\tfrac{4}{3}\right)$$

$$= \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 & 4/3 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 1 & 4/3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 0 & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

<u>OK!</u> $A^{-1} = x_{12}\left(\tfrac{4}{3}\right)^{-1} h_2\left(\tfrac{2}{3}\right)^{-1} h_1(3)^{-1} s_1\left(\tfrac{1}{3}\right)^{-1}$

$$= \begin{pmatrix} 1 & -4/3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

<u>Check:</u>

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2+3 & 1-1 \\ -6+6 & 3-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example M6

$$A=\begin{pmatrix}1&2&1\\-1&-1&1\\0&1&3\end{pmatrix}=s_1(-1)\begin{pmatrix}-1&-1&1\\0&1&2\\0&1&3\end{pmatrix}$$

$$=s_1(-1)s_2(1)\begin{pmatrix}-1&-1&1\\0&1&3\\0&0&-1\end{pmatrix}$$

$$=s_1(-1)s_2(1)h_1(-1)h_3(-1)\begin{pmatrix}1&1&-1\\0&1&3\\0&0&1\end{pmatrix}$$

$$=s_1(-1)s_2(1)h_1(-1)h_3(-1)x_{23}(3)x_{13}(-1)x_{12}(1)$$

Then

$$A^{-1}=x_{12}(1)^{-1}x_{13}(-1)^{-1}x_{23}(3)^{-1}h_3(-1)^{-1}h_1(-1)^{-1}s_2(1)^{-1}s_1(-1)^{-1}$$

$$=\begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&-3\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&1\\0&1&-1\end{pmatrix}\begin{pmatrix}0&1&0\\1&1&0\\0&0&1\end{pmatrix}$$

$$=\begin{pmatrix}1&-1&1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&1&3\\0&0&-1\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&1&-1\end{pmatrix}$$

$$=\begin{pmatrix}-1&-1&-4\\0&1&3\\0&0&-1\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&1&-1\end{pmatrix}=\begin{pmatrix}-4&-5&3\\3&3&-2\\-1&-1&1\end{pmatrix}$$

Check:

$$\begin{pmatrix}1&2&1\\-1&-1&1\\0&1&3\end{pmatrix}\begin{pmatrix}-4&-5&3\\3&3&-2\\-1&-1&1\end{pmatrix}=\begin{pmatrix}-4+6-1&-5+6-1&3-4+1\\4-3-1&5-3-1&-3+2+1\\0+3-3&0+3-3&0-2+3\end{pmatrix}$$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

Example M10

$$A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{pmatrix} = s_2(0) \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

$$= s_2(0)\, \cancel{s_1}(1) \begin{pmatrix} 1 & -3 & 0 & 5 \\ 0 & 2 & 2 & -4 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$

$$= s_2(0)\, s_1(1)\, s_2(2) \begin{pmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= s_2(0)\, s_1(1)\, s_2(2)\, x_{12}(-3) \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since this last right hand factor has a row of 0s then A is not invertible

$$A = s_2(0)\, s_1(1)\, s_2(2)\, x_{12}(-3) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= s_2(0)\, s_1(1)\, s_2(2)\, x_{12}(-3) \cdot 1_2'\, x_{24}(-2)\, x_{14}(5)$$

The <u>rank</u> of A is 2 in this example.