THE UNIVERSITY OF MELBOURNE
SCHOOL OF MATHEMATICS AND STATISTICS

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Arun Ram: Additional Slides

These slides have been made by Arun Ram, in preparation for teaching of the summer session of MAST10007 Linear Algebra at University of Melbourne in 2026. The template is from the University of Melbourne School of Mathematics and Statistics slide deck which was produced by members of the School including, in particular, huge developments by Craig Hodgson and Christine Mangelsdorf.

Root matrices.

$$x_{12}(c) = \begin{pmatrix} 1 & c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{13}(c) = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{23}(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

Diagonal generators.

$$h_1(d) = \begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; h_2(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; h_3(d) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$$

Row reducers.

$$s_1(c) = \begin{pmatrix} c & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Inverses of Root matrices. $x_{12}(c)^{-1} = \begin{pmatrix} 1 & -c & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$x_{13}(c)^{-1} = \begin{pmatrix} 1 & 0 & -c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{23}(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

Inverses of Diagonal generators. $h_1(d)^{-1} = \begin{pmatrix} d^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

$$h_2(d)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; h_3(d)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d^{-1} \end{pmatrix}$$

Inverses of Row reducers.

$$s_1(c)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -c & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad s_2(c)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -c \end{pmatrix}$$

Definition (Invertible matrices)

Let $n \in \mathbb{Z}_{>0}$. *The set of invertible $n \times n$ matrices* is

$$GL_n(\mathbb{Q}) = \left\{ A \in M_{n\times n}(\mathbb{Q}) \;\middle|\; \begin{array}{l} \text{there exists } A^{-1} \in M_{n\times n}(\mathbb{Q}) \\ \text{such that } AA^{-1} = 1 \text{ and } A^{-1}A = 1. \end{array} \right\}$$

Theorem

Let $A, B \in GL_n(\mathbb{Q})$. Then $AB \in GL_n(\mathbb{Q})$ and

$$(AB)^{-1} = B^{-1}A^{-1}.$$

This is because, by associativity,

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1} \cdot 1 \cdot B = B^{-1}B = 1,$$

and

$$(AB)(B^{-1}A^{-1}) = A^{-1}(B^{-1}B)A = A^{-1} \cdot 1 \cdot A = A^{-1}A = 1.$$

Theorem (Greedy normal form for invertible matrices)

Let $n \in \mathbb{Z}_{>0}$. Let $A \in GL_n(\mathbb{Q})$. The greedy normal form gives

$$A = (\text{product of } s_i(c)s) \cdot (\text{product of } h_i(d)s) \cdot (\text{product of } x_{ij}(c)s)$$

This means that if we want to find A^{-1} we can factor A into factors and make it easy to find the inverse of each factor (in reverse order).

Example M8. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = s_1(\tfrac{1}{3}) \begin{pmatrix} 3 & 4 \\ 0 & \frac{2}{3} \end{pmatrix} = s_1(\tfrac{1}{3}) h_1(3) h_2(\tfrac{2}{3}) \begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{pmatrix}$$
$$= s_1(\tfrac{1}{3}) h_1(3) h_2(\tfrac{2}{3}) x_{12}(\tfrac{4}{3}),$$

where

$$s_1(\tfrac{1}{3}) = \begin{pmatrix} \frac{1}{3} & 1 \\ 1 & 0 \end{pmatrix}, \qquad x_{12}(\tfrac{4}{3}) = \begin{pmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{pmatrix},$$

$$h_1(3) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \qquad h_2(\tfrac{2}{3}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}.$$

Then

$$\begin{aligned} A^{-1} &= x_{12}(\tfrac{4}{3})^{-1} h_2(\tfrac{2}{3})^{-1} h_1(3)^{-1} s_1(\tfrac{1}{3})^{-1} \\ &= \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -\frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}. \end{aligned}$$

CHECK:

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example M6. Find the normal form of $A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.

$$\begin{aligned} A &= s_1(-1) \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \\ &= s_1(-1)s_2(1) \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{pmatrix} \\ &= s_1(-1)s_2(1)h_1(-1)h_3(-1) \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \\ &= s_1(-1)s_2(1)h_1(-1)h_3(-1)x_{23}(3)x_{13}(-1)x_{12}(1). \end{aligned}$$

Then

$$
\begin{aligned}
A^{-1} &= x_{12}(1)^{-1}x_{13}(-1)^{-1}x_{23}(3)^{-1}h_3(-1)^{-1}h_1(-1)^{-1}s_2(1)^{-1}s_1(-1)^{-1}\\
&= x_{12}(1)^{-1}x_{13}(-1)^{-1}x_{23}(3)^{-1}h_3(-1)^{-1}h_1(-1)^{-1}\\
&\qquad\cdot \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\\
&= x_{12}(1)^{-1}x_{13}(-1)^{-1}x_{23}(3)^{-1}\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 1 & -1\end{pmatrix}\\
&= \begin{pmatrix}1 & -1 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & -3\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1 & 0\\ 0 & 0 & 1\\ -1 & -1 & 1\end{pmatrix}\\
&= \begin{pmatrix}1 & -1 & 1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}0 & -1 & 0\\ 3 & 3 & -2\\ -1 & -1 & 1\end{pmatrix} = \begin{pmatrix}-4 & -5 & 3\\ 3 & 3 & -2\\ -1 & -1 & 1\end{pmatrix}
\end{aligned}
$$

Example M10. Let $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -2 \\ 1 & -3 & 0 & 5 \end{pmatrix}$. Then

$$A = s_2(0) \begin{pmatrix} 1 & -1 & 2 & 1 \\ 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & -2 \end{pmatrix} = s_2(0)s_1(1) \begin{pmatrix} 1 & -3 & 0 & 5 \\ 0 & 2 & 2 & -4 \\ 0 & 1 & 1 & -2 \end{pmatrix}$$
$$= s_2(0)s_1(1)s_2(2) \begin{pmatrix} 1 & -3 & 0 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$= s_2(0)s_1(1)s_2(2)x_{12}(-3) \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since this last right hand factor has a row of 0s then A is not invertible.

$$A = s_2(0)s_1(1)s_2(2)x_{12}(-3) \cdot 1_2 \cdot x_{23}(1)x_{14}(5)x_{24}(-2).$$

So $\mathrm{rank}(A) = 2$.

Theorem (Greedy normal form for all matrices)

Let $s, t \in \mathbb{Z}_{>0}$. Let E_{ij} be the $t \times s$ matrix with 1 in the (i, j) entry and 0 elswhere. For $r \in \{1, \ldots, \min(s, t)\}$ let

$$1_r = E_{11} + \cdots + E_{rr}.$$

Let $A \in M_{t\times s}(\mathbb{Q})$. The greedy normal form gives

$$A = (\textit{product of } s_i(c)s) \cdot (\textit{product of } h_i(d)s) \cdot (\textit{product of } x_{ij}(c)s) \cdot 1_r \cdot (\textit{product of } s_i(c)s) \cdot (\textit{product of } x_{ij}(c)s).$$

The number r that comes out of the greedy normal form is the *rank* of A. Later the rank of A will be realised as the dimension of the image of A,

$$r = \dim(\mathrm{im}(A)) = \mathrm{rank}(A) \quad \text{is the } \textit{rank} \text{ of } A.$$