Let V be an $\mathbb{C}$-vector space and $\langle , \rangle : V \times V \to \mathbb{C}$ an inner product on V.

Let $u, v \in V$. The vectors u and v are

<u>orthogonal</u> if $\langle u, v\rangle = 0$.

A matrix A is <u>orthogonal</u> if $AA^t = 1$.

An <u>orthogonal sequence</u> in V is a sequence $(b_1, \ldots, b_k)$ in V such that if

if $i, j \in \{1, \ldots, k\}$ and $i \neq j$ then $\langle b_i, b_j\rangle = 0$.

An <u>orthonormal sequence</u> is an orthogonal sequence $(b_1, \ldots, b_k)$ such that

if $i \in \{1, \ldots, k\}$ then $\langle b_i, b_i\rangle = 1$.

An <u>orthonormal basis</u> is an orthonormal sequence that is a basis of V.

Orthonormal bases are really good for computing coordinate vectors and projections.

Example IP 13 (Gram-Schmidt orthogonalisation)

Let $V = \mathbb{R}^3$ with the standard inner product.

Let

$S = \{v_1, v_2, v_3\}$ with $v_1 = \langle 1,1,1 \rangle$
$v_2 = \langle 0,1,1 \rangle$
$v_3 = \langle 0,0,1 \rangle$.

Convert S into an orthonormal basis $\mathcal{B}$

Step 1 Make v_1 into a unit vector.

$$b_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{3}} \langle 1,1,1 \rangle = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

and let $S = \{b_1, v_2, v_3\rangle$.

Step 2 Make v_2 orthogonal to b_1.

Let $u_2 = v_2 - \langle v, b_1 \rangle b_1$

$$= \langle 0,1,1 \rangle - \frac{2}{\sqrt{3}} \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$= \left\langle \frac{-2}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle.$$

and let $S_2 = \{b_1, u_2, v_3\}$.

Step 3. Make u_2 into a unit vector.

$$b_2 = \frac{1}{\|u_2\|} u_2 = \frac{1}{\sqrt{6}/3} \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle = \left\langle \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle.$$

and let $S_3 = \{b_1, b_2, v_3\}$.

Step 4 Make v_3 orthogonal to b_1 and b_2.

Let

$$u_3 = v_3 - \langle v_3, b_1 \rangle b_1 - \langle v_3, b_2 \rangle b_2$$

$$= \langle 0,0,1 \rangle - \frac{1}{\sqrt{3}} \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle - \frac{1}{\sqrt{6}} \left\langle \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$= \left\langle -\frac{1}{3} + \frac{2}{6}, -\frac{1}{3} - \frac{1}{6}, 1 - \frac{1}{3} - \frac{1}{6} \right\rangle = \left\langle 0, -\frac{1}{2}, \frac{1}{2} \right\rangle$$

Step 5 Make u_3 into a unit vector.

Let

$$b_3 = \frac{1}{\|u_3\|} u_3 = \frac{1}{\sqrt{2/4}} \left\langle 0, -\frac{1}{2}, \frac{1}{2} \right\rangle = \left\langle 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Then

$$B = \{ b_1, b_2, b_3 \}$$ is an orthonormal set.

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Lin. Alg Lect 23
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Example IP 9 and 10 and 11 Let

$\langle\ \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be given by

$$\langle (u_1,u_2,u_3),(v_1,v_2,v_3)\rangle = u_1v_1 + 2u_2v_2 + u_3v_3.$$

Let $S = \{(1,1,1),(1,-1,1),(1,0,-1)\}$. Then

$$\langle (1,1,1),(1,-1,1)\rangle = 1-2+1 = 0,$$
$$\langle (1,1,1),(1,0,-1)\rangle = 1+0-1 = 0,$$
$$\langle (1,-1,1),(1,0,-1)\rangle = 1+0+-1 = 0.$$

So S is an orthogonal sequence in $\mathbb{R}^3$ with respect to $\langle\ ,\ \rangle$.

Let $s_1 = \frac{1}{\|u_1\|}u_1$ where $u_1 = (1,1,1)$

$s_2 = \frac{1}{\|u_2\|}u_2$ where $u_2 = (1,-1,1)$

$s_3 = \frac{1}{\|u_3\|}u_3$ where $u_3 = (1,0,-1)$.

Then

$s_1 = \frac{1}{2}(1,1,1)$ since $\langle u_1,u_1\rangle = 1+2+1 = 4$

$s_2 = \frac{1}{2}(1,-1,1)$ since $\langle u_2,u_2\rangle = 1+2+1 = 4$

$s_3 = \frac{1}{\sqrt{2}}(1,0,-1)$ since $\langle u_3,u_3\rangle = 1+0+1 = 2$.

Then (s_1,s_2,s_3) is an orthonormal sequence in $\mathbb{R}^3$ with respect to $\langle\ ,\ \rangle$.

Let $x=(1,1,-1)$. Then

$\langle x,b_1\rangle = \langle(1,1,-1),(\frac{1}{2},\frac{1}{2},\frac{1}{2})\rangle = \frac{1}{2}+1-\frac{1}{2}=1$

$\langle x,b_2\rangle = \langle(1,1,-1),(\frac{1}{2},\frac{-1}{2},\frac{1}{2})\rangle = \frac{1}{2}-1\frac{-1}{2}=-1$

$\langle x,b_2\rangle = \langle(1,1,-1),(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}})\rangle = \frac{1}{\sqrt{2}}+0+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}$.

and

$\langle x,b_1\rangle b_1+\langle x,b_2\rangle b_2+\langle x,b_3\rangle b_3$

$= 1\cdot b_1+(-1)\cdot b_2+\sqrt{2}\,b_3$

$= 1\cdot(\frac{1}{2},\frac{1}{2},\frac{1}{2})-(\frac{1}{2},\frac{-1}{2},\frac{1}{2})+\sqrt{2}(\frac{1}{\sqrt{2}},0,\frac{-1}{\sqrt{2}})$

$= (\frac{1}{2}-\frac{1}{2}+1,\ \frac{1}{2}+\frac{1}{2}+0,\ \frac{1}{2}-\frac{1}{2}-1) = (1,1,-1)=x$.

<u>Example</u> <u>IP12</u> Let $V=\mathbb{R}^3$ and

$W=\{(x,y,z)\in\mathbb{R}^3 \mid x+y+z=0\}$

Then

$B=\{b_1,b_2\}$ with $b_1=(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}},0)$

$b_2=(\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{-2}{\sqrt{6}})$

is an orthonormal basis of W with respect to the standard inner product on $\mathbb{R}^3$.

Let $x = \langle 1, 2, 3 \rangle$. Then

$$\mathrm{proj}_W(x) = \langle x \mid b_1 \rangle b_1 + \langle x \mid b_2 \rangle b_2$$

$$= \langle 1,2,3 \mid \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0 \rangle \cdot \langle \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0 \rangle$$

$$+ \langle 1,2,3 \mid \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}} \rangle \cdot \langle \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}} \rangle$$

$$= \left(\tfrac{1}{\sqrt{2}} - \tfrac{2}{\sqrt{2}} \right) \cdot \langle \tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0 \rangle$$

$$+ \left(\tfrac{1}{\sqrt{6}} + \tfrac{2}{\sqrt{6}} - \tfrac{6}{\sqrt{6}} \right) \cdot \langle \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}} \rangle$$

$$= \langle \tfrac{-1}{2}, \tfrac{1}{2}, 0 \rangle + \langle \tfrac{-3}{6}, \tfrac{-3}{6}, \tfrac{6}{6} \rangle$$

$$= \langle -1, 0, 1 \rangle.$$

The shortest distance from x to W is

$$\| x - \mathrm{proj}_W(x) \| = \| \langle 1,2,3 \rangle - \langle -1, 0, 1 \rangle \|$$

$$= \| \langle 2, 2, 2 \rangle \| = \sqrt{4+4+4} = 2\sqrt{3}.$$