Lecture 26: Application of diagonalization to dynamics

Theorem (Diagonalization.)

Let $A \in M_n(\mathbb{F})$. The matrix A has n linearly independent eigenvectors $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{F}^n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if

$$A = PDP^{-1}$$

where,

$$P = \begin{pmatrix} | & & | \\ \mathbf{p}_1 & \cdots & \mathbf{p}_n \\ | & & | \end{pmatrix} \quad \text{and} \quad D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

so that $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are the columns of P and D is the diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.

Example EV13.

$$\text{If} \qquad D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{then} \qquad D^{10} = \begin{pmatrix} (-4)^{10} & 0 & 0 \\ 0 & 3^{10} & 0 \\ 0 & 0 & 2^{10} \end{pmatrix}.$$

Example EV14. If $A = PDP^{-1}$ then

$$\begin{aligned} A^3 &= A \cdot A \cdot A = (PDP^{-1})(PDP^{-1})(PDP^{-1}) = PDP^{-1}PDP^{-1}PDP^{-1} \\ &= PD \cdot D \cdot DP^{-1} = PD^3P^{-1}, \end{aligned}$$

and, similarly, if $k \in \mathbb{Z}$ then

$$A^k = PD^kP^{-1}.$$

Let

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}, \qquad P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}.$$

Then

$$P^{-1} = \tfrac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad A = PDP^{-1}.$$

So

$$\begin{aligned} A^k &= PD^kP^{-1} = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (-1)^k & 0 \\ 0 & 3^k \end{pmatrix} \tfrac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} (-1)^k \cdot 2 & 3^k \cdot 2 \\ (-1)^k & 3^k \end{pmatrix} \tfrac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix} \\ &= \tfrac{1}{4} \begin{pmatrix} 2((-1)^k + 3^k) & 4((-1)^{k+1} + 3^k) \\ (-1)^{k+1} + 3^k & 2((-1)^k + 3^k) \end{pmatrix} \end{aligned}$$

Example EV15. Let

$$x_n = \begin{pmatrix} r_n \\ p_n \\ w_n \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

and define an evolution process by

$$x_{n+1} = Tx_n.$$

This is the *Markov chain* defined by T. Since $T = PDP^{-1}$, where

$$P = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; P^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

then *the stationary state of the process on* $\mathbb{R}^3$ *defined by* T is

$$\lim_{n\to\infty} T^n x_0 = \lim_{n\to\infty} PD^n P^{-1} x_0 = \lim_{n\to\infty} P \begin{pmatrix} 1^n & 0 & 0 \\ 0 & \left(\frac{1}{2}\right)^n & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1}$$

$$= P \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} P^{-1} x_0 = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x_0$$

$$= \begin{pmatrix} \frac{1}{4}(r_0 + p_0 + w_0) \\ \frac{1}{2}(r_0 + p_0 + w_0) \\ \frac{1}{4}(r_0 + p_0 + w_0) \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$