Coordinate vector of v with respect to B

$$[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{if} \quad v = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n.$$

Matrix of a linear transformation $T: V \to W$ with respect to a basis B of V and a basis D of W is

$$[T]_{DB} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \quad \text{if} \quad \begin{aligned} T(b_1) &= a_{11} d_1 + \cdots + a_{n1} d_n \\ &\vdots \\ T(b_n) &= a_{1n} d_1 + \cdots + a_{nn} d_n \end{aligned}$$

Gram matrix of an inner product $\langle\, , \rangle : V \times V \to \mathbb{C}$ with respect to a basis B of V

$$\diamond_B = \begin{pmatrix} \langle b_1, b_1 \rangle & \cdots & \langle b_1, b_n \rangle \\ \langle b_2, b_1 \rangle & \cdots & \langle b_2, b_n \rangle \\ \vdots & & \\ \langle b_n, b_1 \rangle & \cdots & \langle b_n, b_n \rangle \end{pmatrix}.$$

Since $\langle b_j, b_i \rangle = \overline{\langle b_i, b_j \rangle}$ then $\overline{\diamond_B}^T = \diamond_B$.

So $\diamond_B$ is Hermitian

<u>Example IP4</u> $\mathbb{R}[x]_{\le 2} = \{a_0 + a_1 x + a_2 x^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$ has standard basis $\{1, x, x^2\} = B$ and standard inner product $\langle p | q \rangle = \int_0^1 pq\,dx$.

Then

$\langle 1|1\rangle = \int_0^1 1\,dx = x\Big]_0^1 = 1 - 0 = 1,$

$\langle 1|x\rangle = \int_0^1 x\,dx = \frac{1}{2}x^2\Big]_0^1 = \frac{1}{2}$ and $\langle x|1\rangle = \frac{1}{2}$

$\langle 1|x^2\rangle = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}$ and $\langle x^2|1\rangle = \frac{1}{3}$.

$\langle x|x\rangle = \int_0^1 x^2\,dx = \frac{1}{3}x^3\Big]_0^1 = \frac{1}{3}$.

$\langle x|x^2\rangle = \int_0^1 x^3\,dx = \frac{1}{4}x^4\Big]_0^1 = \frac{1}{4}$ and $\langle x^2|x\rangle = \frac{1}{4}$

$\langle x^2, x^2\rangle = \int x^4\,dx = \frac{1}{5}x^5\Big]_0^1 = \frac{1}{5}$

The Gram matrix of $\langle\,|\,\rangle$ with respect to the basis B is

$$A_B = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

If $u = 7+3x+2x^2$ and $v = 5+x^2$ then

$$[u]_B = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \text{ and } [v]_B = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

Then

$$\langle u|v\rangle = [u]_B^T A [v]_B = (7\ 3\ 2)\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

$$= (7\ 3\ 2)\begin{pmatrix} 5+\frac{1}{3} \\ \frac{5}{2}+\frac{1}{4} \\ \frac{5}{3}+\frac{1}{5} \end{pmatrix} = 35+\frac{7}{3}+\frac{15}{2}+\frac{3}{4}+\frac{10}{3}+\frac{2}{5}.$$

and this checks with

$$\langle u|v\rangle = \int_0^1 uv\,dx = \int_0^1 (7+3x+2x^2)(5+x^2)\,dx$$

$$= \int_0^1 (35+7x^2+15x+3x^3+10x^2+2x^4)\,dx$$

$$= 35x+\frac{7}{3}x^3+\frac{15}{2}x^2+\frac{3}{4}x^4+\frac{10}{3}x^3+\frac{2x^5}{5}\Big]_0^1$$

$$= 35+\frac{7}{3}+\frac{15}{2}+\frac{3}{4}+\frac{10}{3}+\frac{2}{5}$$

Let V be a $\mathbb{C}$-vector space with an inner product $\langle\,,\rangle : V \times V \to \mathbb{C}$.

Let W be a subspace of V.

Let $(b_1, b_2, \ldots, b_k)$ be an orthonormal basis of W.

Let $x \in V$.

The <u>projection of x onto W</u> is

$$\mathrm{proj}_W(x) = \langle x, b_1\rangle b_1 + \cdots + \langle x, b_k\rangle b_k.$$

<u>Special cases:</u>

(1) If $W = V$ then $\mathrm{proj}_V(x) = x$ and

$$x = \langle x, b_1\rangle b_1 + \cdots + \langle x, b_n\rangle b_n$$

if $(b_1, \ldots, b_n)$ is an orthonormal basis of V.

(2) If $\dim(W) = 1$ then $W = \mathrm{span}\{\vec{u}\}$ and $b_1 = \frac{1}{\|u\|}\vec{u}$ gives $\{b_1\}$ is an orthonormal basis of W. Then

$$\mathrm{proj}_W(x) = \mathrm{proj}_u(x) = \langle x, b_1\rangle b_1 = \frac{\langle x, u\rangle}{\|u\|}\frac{\vec{u}}{\|u\|}$$

$$= \frac{\langle x, u\rangle}{\langle u, u\rangle}\vec{u}.$$