Lecture 25: Projections and orthogonalisation

Definition (Orthogonal and orthonormal sequences.)

Let V be an $\mathbb{F}$-vector space with an inner product $\langle , \rangle \colon V \times V \to \mathbb{F}$.

Let $u, v \in V$. The vectors u and v are

$$\textit{orthogonal} \quad \text{if } \langle u, v \rangle = 0.$$

An *orthogonal sequence* is a sequence $(b_1, \ldots, b_k)$ of vectors in V such that

$$\text{if } i, j \in \{1, \ldots, k\} \text{ and } i \neq j \quad \text{then} \quad \langle b_i, b_j \rangle = 0.$$

An *orthonormal sequence* is an orthogonal sequence $(b_1, \ldots, b_k)$ such that

$$\text{if } i \in \{1, \ldots, k\} \text{ then } \langle b_i, b_i \rangle = 1.$$

An *ordered orthonormal basis of V* is an orthonormal sequence $(b_1, \ldots, b_k)$ in V such that B is a basis of V.

Proposition

Assume $B = (b_1, \ldots, b_n)$ *is an ordered orthonormal basis of* V *and* $x \in V$. *Then*

$$x = \langle x, b_1 \rangle b_1 + \cdots + \langle x, b_n \rangle b_n.$$

Definition (Orthogonal projections.)

Let W be a subspace of V. Let $\{b_1, \ldots, b_k\}$ be an orthonormal basis of W. Let $x \in V$. The *orthogonal projection of x onto W* is

$$\text{proj}_W(x) = \langle x, b_1 \rangle b_1 + \cdots + \langle x, b_k \rangle b_k.$$

Example IP9,10&11 Let $\langle , \rangle \colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be given by

$$\langle (u_1, u_2, u_3), (v_1, v_2, v_3) \rangle = u_1 v_1 + 2 u_2 v_2 + u_3 v_3.$$

Let

$$S = \{(1, 1, 1), (1 - 1, 1), (1, 0 - 1)\}.$$

Then

$$\begin{aligned}
\langle (1, 1, 1), (1, -1, 1) \rangle &= 1 - 2 + 1 = 0, \\
\langle (1, 1, 1), (1, 0, -1) \rangle &= 1 + 0 - 1 = 0, \\
\langle (1, -1, 1), (1, 0, -1) \rangle &= 1 + 0 - 1 = 0,
\end{aligned}$$

So S is an orthogonal sequence in $\mathbb{R}^3$ with respect to $\langle , \rangle$.

Let

$$
\begin{aligned}
b_1 &= \frac{1}{\|u\|} u, && \text{where} \quad u = (1, 1, 1), \\
b_2 &= \frac{1}{\|v\|} v, && \text{where} \quad v = (1, -1, 1), \\
b_3 &= \frac{1}{\|w\|} w, && \text{where} \quad w = (1, 0, 01).
\end{aligned}
$$

Then

$$
\begin{aligned}
b_1 &= \tfrac{1}{2}(1, 1, 1) \quad \text{since} && \langle (1,1,1), (1,1,1) \rangle = 4, \\
b_2 &= \tfrac{1}{2}(1, -1, 1) \quad \text{since} && \langle (1,-1,1), (1,-1,1) \rangle = 4, \\
b_2 &= \tfrac{1}{\sqrt{2}}(1, 0, -1) \quad \text{since} && \langle (1,0,-1), (1,0,-1) \rangle = 2,
\end{aligned}
$$

and $\{b_1, b_2, b_3\}$ is an orthonormal sequence in $\mathbb{R}^3$ with respect to $\langle , \rangle$.

Let $x = |1, 1, -1\rangle$ and $c_2, c_2, c_3 \in \mathbb{R}$ such that

$$x = c_1 b_1 + c_2 b_2 + c_3 b_3.$$

Then

$$\begin{aligned}
c_1 &= c_1 \langle b_1, b_1 \rangle + 0 + 0 = \langle c_1 b_1 + c_2 b_2 + c_3 b_3, b_1 \rangle = \langle x, b_1 \rangle \\
&= \langle (1, 1, -1), \tfrac{1}{2}(1, 1, 1) \rangle = \tfrac{1}{2}(1 + 2 - 1), \\
c_2 &= c_2 \langle b_2, b_2 \rangle = \langle c_1 b_1 + c_2 b_2 + c_3 b_3, b_2 \rangle = \langle x, b_2 \rangle \\
&= \langle (1, 1, -1), \tfrac{1}{2}(1, -1, 1) \rangle = \tfrac{1}{2}(1 - 2 - 1), \\
c_3 &= \langle x, b_3 \rangle = \langle (1, 1, -1), \tfrac{1}{\sqrt{2}}(1, 0, -1) \rangle = \tfrac{1}{\sqrt{2}}(1 + 0 + 1) = \tfrac{2}{\sqrt{2}} = \sqrt{2}.
\end{aligned}$$

So x is written as a linear combination of the basis elements in the form

$$\begin{aligned}
x = (1, 1, -1) &= \langle x, b_1 \rangle b_1 + \langle x, b_2 \rangle b_2 + \langle x, b_3 \rangle b_3 \\
&= 1 \cdot b_1 + (-1) \cdot b_2 + \sqrt{2} b_3 \\
&= (1, 1, 1) - (1, -1, 1) + \sqrt{2}(1, 0, -1).
\end{aligned}$$

Example IPA3. Let $V = \mathbb{R}^n$ and let $u \in V$ with $u \neq 0$. Let

$$W = \mathbb{R}\text{-span}\{u\} = \{au \mid a \in \mathbb{R}\}.$$

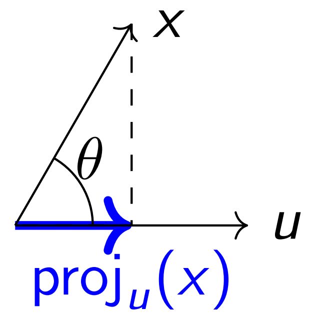
Then W is a 1-dimensional subspace of V. Let

$$b_1 = \frac{1}{\|u\|}u.$$

Then $\{b_1\}$ is an orthonormal basis of W.
Let $x \in V$. Then

$$\text{proj}_W(x) = \langle x, b_1\rangle b_1 = \langle x, \frac{1}{\|u\|}u\rangle \frac{1}{\|u\|}u$$

$$= \frac{\langle x, u\rangle}{\|u\|^2}u = \text{proj}_u(x).$$



Example IP12. Let $V = \mathbb{R}^3$ and let

$$W = \{|x, y, z\rangle \in \mathbb{R}^3 \mid x + y + z = 0\}.$$

The set

$$\{b_1, b_2\} = \left\{\tfrac{1}{\sqrt{2}}|1, -1, 0\rangle, \tfrac{1}{\sqrt{6}}|1, 1, -2\rangle\right\}$$

is an orthonormal basis of W with respect to the standard inner product on $\mathbb{R}^3$.

Let $x = |1, 2, 3\rangle$. Then

$$\begin{aligned}
\mathrm{proj}_W(x) &= \langle x|b_1\rangle b_1 + \langle x|b_2\rangle b_2 \\
&= \langle 1, 2, 3|\tfrac{1}{\sqrt{2}}, \tfrac{-1}{\sqrt{2}}, 0\rangle \cdot \tfrac{1}{\sqrt{2}}|1, -1, 0\rangle \\
&\quad + \langle 1, 2, 3|\tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{-2}{\sqrt{6}}\rangle \cdot \tfrac{1}{\sqrt{6}}|1, 1, -2\rangle \\
&= \left(\tfrac{1}{\sqrt{2}} - \tfrac{2}{\sqrt{2}}\right)\tfrac{1}{\sqrt{2}}|1, -1, 0\rangle + \tfrac{1}{6}(1 + 2 - 6)|1, 1, -2)\rangle \\
&= |\tfrac{-1}{2}, \tfrac{1}{2}, 0\rangle + |\tfrac{-1}{2}, \tfrac{-1}{2}, 1) = |-1, 0, 1\rangle.
\end{aligned}$$

The shortest distance from x to W is

$$\begin{aligned}
\|x - \mathrm{proj}_W(x)\| &= \|\, |1, 2, 3\rangle - |-1, 0, 1\rangle \,\| \\
&= \|\, |2, 2, 2\rangle \,\| = \sqrt{4 + 4 + 4} = 2\sqrt{3}.
\end{aligned}$$

Example IP13. *(The Gram-Schmidt process of orthogonalization)*
Let $V = \mathbb{R}^3$ with the standard inner product. Let $S = \{v_1, v_2, v_3\}$ with

$$v_1 = |1,1,1\rangle, \quad v_2 = |0,1,1\rangle, \quad v_1 = |0,0,1\rangle.$$

Convert S into an orthonormal basis B.

Step 1. Make v_1 into a unit vector. Let

$$b_1 = \frac{1}{\|v_1\|} v_1 = \tfrac{1}{\sqrt{3}}|1,1,1\rangle = |\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\rangle$$

and let $S = \{b_1, v_2, v_3\}$.

Step 2. Make v_2 orthogonal to b_1. Let

$$\begin{aligned} u_2 &= v_2 - \langle v, b_1\rangle b_1 \\ &= |0,1,1\rangle - \tfrac{2}{\sqrt{3}}|\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\rangle \\ &= |\tfrac{-2}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\rangle \end{aligned}$$

and let $S_2 = \{b_1, u_2, v_3\}$.

Step 3. Make u_2 into a unit vector. Let

$$b_2 = \frac{1}{\|u_2\|} u_2 = \frac{1}{\sqrt{6}/3} |\tfrac{-2}{3}, \tfrac{1}{3}, \tfrac{1}{3}\rangle = |\tfrac{-2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\rangle$$

and let $S_3 = \{b_1, b_2, v_3\}$.

Step 4. Make v_3 orthogonal to b_1 and b_2. Let

$$\begin{aligned} u_3 &= v_3 - \langle v_3, b_1\rangle b_1 - \langle v_3, b_2\rangle b_2 \\ &= |0, 0, 1\rangle - \tfrac{1}{\sqrt{3}} |\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\rangle - \tfrac{1}{\sqrt{6}} |\tfrac{-2}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}, \tfrac{1}{\sqrt{6}}\rangle \\ &= |\tfrac{-1}{3} + \tfrac{2}{6}, \tfrac{-1}{3} - \tfrac{1}{6}, 1 - \tfrac{1}{3} - \tfrac{1}{6}\rangle = |0, \tfrac{-1}{2}, \tfrac{1}{2}\rangle. \end{aligned}$$

Step 5. Make u_3 into a unit vector. Let

$$b_3 = \frac{1}{\|u_3\|} u_3 = \frac{1}{\sqrt{2/4}} |0, \tfrac{-1}{2}, \tfrac{1}{2}\rangle = |0, \tfrac{-1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle.$$

Then

$$B = \{b_1, b_2, b_3\} \quad \text{is an orthonormal set.}$$

Proposition

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and let V be an $\mathbb{F}$-vector space with an inner product $\langle , \rangle \colon V \times V \to \mathbb{F}$. If S is an orthonormal set then S is linearly independent.

Proof. Assume $S = \{v_1, v_2, \ldots, v_k\}$ is an orthonormal set.
To show: If $c_1, \ldots, c_k \in \mathbb{F}$ and $c_1 v_1 + \cdots + c_k v_k = 0$ then $c_1 = 0,\ c_2 = 0, \ldots, c_k = 0$.
Assume $c_1, \ldots, c_k \in \mathbb{F}$ and $c_1 v_1 + \cdots + c_k v_k = 0$.
To show: If $j \in \{1, \ldots, k\}$ then $c_j = 0$.
Assume $j \in \{1, \ldots, k\}$. To show: $0 = c_j$.

$$\begin{aligned}
0 &= \langle c_1 v_2 + \cdots + c_k v_k, v_j\rangle \\
&= c_1\langle v_1, v_j\rangle + \cdots + c_{j-1}\langle v_{j-1}, v_j\rangle + c_j\langle v_j, v_j\rangle \\
&\quad + c_{j+1}\langle v_{j+1}, v_j\rangle + \cdots + c_k\langle v_k, v_j\rangle \\
&= c_1 \cdot 0 + \cdots + c_{j-1} \cdot 0 + c_j\langle v_j, v_j\rangle + c_{j+1} \cdot 0 + \cdots + c_k \cdot 0 \\
&= c_j\langle v_j, v_j\rangle.
\end{aligned}$$

Since $0 = c_j\langle v_j, v_j\rangle$ and $\langle v_j, v_j\rangle \neq 0$ then $c_j = 0$.
So S is linearly independent.