A matrix $A \in M_{n\times n}(\mathbb{C})$ is <u>Hermitian</u> if $A = \bar{A}^T$

A matrix $A \in M_{n\times n}(\mathbb{C})$ is <u>symmetric</u> if $A = A^T$

A matrix $A \in M_{n\times n}(\mathbb{C})$ is <u>orthogonal</u> if $AA^T = I$.

A matrix $A \in M_{n\times n}(\mathbb{C})$ is <u>unitary</u> if $A\bar{A}^T = I$.

<u>Examples</u> (1) $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ has $A^T = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$

and $\bar{A}^T = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. So A is Hermitian

(2) $B = \begin{pmatrix} +i & 0 \\ 0 & +i \end{pmatrix}$ has $B^T = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ and $\bar{B}^T = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}$

So B is <u>not</u> Hermitian.

(3) $A = \begin{pmatrix} 12 & 19 \\ 19 & 12 \end{pmatrix}$ is symmetric.

$B = \begin{pmatrix} 12 & 17 \\ 19 & 12 \end{pmatrix}$ is not symmetric.

(4) $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ has $Q^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

and

$$QQ^T = \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So Q is orthogonal.

Example IP 23 Let $A = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$. Then

$A^T = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $\bar{A}^T = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$ and A is Hermitian

Find eigenvalues and orthonormal eigenvectors in $\mathbb{C}^2$.

To do this find $\ker(A-\lambda)$

$$A-\lambda = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & i \\ -i & 1-\lambda \end{pmatrix}.$$

Left multiply $A-\lambda$ by

$$\begin{pmatrix} 0 & 1 \\ 1 & -\frac{(1-\lambda)}{-i} \end{pmatrix} \text{ to get } \begin{pmatrix} -i & 1-\lambda \\ 0 & i + \frac{(1-\lambda)^2}{i} \end{pmatrix}$$

and $i + \frac{(1-\lambda)^2}{i} = i - i(1-\lambda)^2 = i - i(1-2\lambda+\lambda^2)$

$$= 2i\lambda - i\lambda^2 = i\lambda(2-\lambda).$$

So we get a row of 0's if $\lambda = 0$ or $\lambda = 2$.

Case 1: $\lambda = 0$ then

$$\ker(A-\lambda) = \ker\begin{pmatrix} -i & 1-0 \\ 0 & 0 \end{pmatrix} = \ker\begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, \begin{matrix} -ix_1 + x_2 = 0 \\ 0 = 0 \end{matrix} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2 \,\middle|\, \begin{matrix} x_2 = ix_1 \\ x_1 = x_1 \end{matrix} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ ix_1 \end{pmatrix} \,\middle|\, x_1 \in \mathbb{C} \right\} = \left\{ x_1 \begin{pmatrix} 1 \\ i \end{pmatrix} \,\middle|\, x_1 \in \mathbb{C} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ i \end{pmatrix} \right\}$$

$= \operatorname{span}\left\{ \begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix} \right\}$ with $\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$ a unit vector.

Case 2: $\lambda = 2$.

$$\ker(A-\lambda) = \ker\begin{pmatrix} -i & 1-2 \\ 0 & 0 \end{pmatrix} = \ker\begin{pmatrix} -i & -1 \\ 0 & 0 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \begin{matrix} -ix_1 - x_2 = 0 \\ 0 = 0 \end{matrix} \right\} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{C}^2 \;\middle|\; \begin{matrix} x_2 = -i x_1 \\ x_2 = x_2 \end{matrix} \right\}$$

$$= \left\{ x_1 \begin{pmatrix} 1 \\ -i \end{pmatrix} \;\middle|\; x_1 \in \mathbb{C} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix} 1/\sqrt{2} \\ -\frac{i}{\sqrt{2}} \end{pmatrix} \right\}$$

with $\begin{pmatrix} -i/\sqrt{2} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ a unit vector.

Then $\begin{pmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ -\frac{i}{\sqrt{2}} \end{pmatrix}$ are orthogonal vectors.

Let $P = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ i/\sqrt{2} & -\frac{i}{\sqrt{2}} \end{pmatrix}$. Then $P^{-1} = \bar{P}^t = \begin{pmatrix} \frac{1}{\sqrt{2}} & -i/\sqrt{2} \\ \frac{1}{\sqrt{2}} & +\frac{i}{\sqrt{2}} \end{pmatrix}$

and $\bar{P}^t A P = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ and P is unitary.

The standard inner product on $\mathbb{C}^n$ is $\langle , \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ given by

$$\langle u_1, \ldots, u_n | v_1, \ldots, v_n \rangle = u_1 \bar{v}_1 + \cdots + u_n \bar{v}_n$$

$$= (u_1 \cdots u_n) \begin{pmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{pmatrix} = u^T \bar{v}.$$

Most all inner products look like

$\langle u, v \rangle = u^T A v$ where A is an $n \times n$ matrix such that $A = \bar{A}^T$ (Hermitian)

The standard inner product on polynomials is $\mathbb{C}[x]_{\le n} = \{ a_0 + a_1 x + \cdots + a_n x^n \mid a_0, a_1, \ldots, a_n \in \mathbb{C} \}$ is

$$\langle a_0 + a_1 x + \cdots + a_n x^n | c_0 + c_1 x + \cdots + c_n x_n \rangle$$

$$= \int_0^1 (a_0 + a_1 x + \cdots + a_n x^n)(\bar{c}_0 + \bar{c}_1 x_1 + \cdots + \bar{c}_n x_n^n)\, dx$$

In other words, if $p(x), q(x) \in \mathbb{C}[x]_{\le n}$ then

$$\langle p(x), q(x) \rangle = \int_0^1 p(x) \overline{q(x)}\, dx.$$

With inner products, we have

<u>Lengths</u> $\|u\| = \sqrt{\langle u, u\rangle}$

<u>distances</u> $d(u,v) = \|v-u\| = \sqrt{\langle v-u, v-u\rangle}$

<u>angles</u> $\cos(\theta(u,v)) = \dfrac{Re(\langle u,v\rangle)}{\|u\|\cdot\|v\|}$.

and <u>projections</u>. For projections we will need <u>orthogonalisation</u>.

<u>Example IP7</u> Let $u=1$ and $v=x$.

Then $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\int_0^1 x\cdot x\,dx} = \sqrt{\frac{1}{3}x^3\Big]_0^1}$

$= \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$.

and

$d(1,x) = \sqrt{\langle x-1, x-1\rangle} = \sqrt{\int_0^1 (x^2-2x+1)\,dx} = \sqrt{\frac{1}{3}x^3 - x^2 + x\Big]_0^1}$

$= \sqrt{\frac{1}{3} - 1 + 1} = \frac{1}{\sqrt{3}}$

$\cos(\theta(1,x)) = \dfrac{Re(\langle 1, x\rangle)}{\|1\|\cdot\|x\|} = \dfrac{Re\left(\int_0^1 x\,dx\right)}{\sqrt{\int_0^1 dx}\sqrt{\int_0^1 x^2\,dx}} = \dfrac{Re\left(\frac{1}{2}x^2\Big]_0^1\right)}{\sqrt{x\Big]_0^1}\sqrt{\frac{1}{3}x^3\Big]_0^1}}$

$= \dfrac{\frac{1}{2}}{1\cdot(1/\sqrt{3})} = \frac{\sqrt{3}}{2}$. So $\theta(1,x) = \pi/6$