Lecture 24: Gram matrices, orthogonality and projections

Definition (Gram matrix.)

Let V be an $\mathbb{F}$-vector space with inner product $\langle , \rangle V \times V \to \mathbb{F}$.
Let $B = \{b_1, \ldots, b_n\}$ be a basis of V.
The *Gram matrix of $\langle , \rangle$ with respect to B* is the matrix

$$A = \begin{pmatrix} \langle b_1, b_1 \rangle & \langle b_1, b_2 \rangle & \cdots & \langle b_1, b_n \rangle \\ \langle b_2, b_1 \rangle & \langle b_2, b_2 \rangle & \cdots & \langle b_2, b_n \rangle \\ \vdots & & & \vdots \\ \langle b_n, b_1 \rangle & \langle b_n, b_2 \rangle & \cdots & \langle b_n, b_n \rangle \end{pmatrix}$$

In other words, the (i, j) entry of the Gram matrix A of $\langle , \rangle$ with respect to the basis B is

$$A_{ij} = \langle b_i, b_j \rangle.$$

Example IPA2. Let V be a $\mathbb{F}$-vector space with an inner product $\langle , \rangle : V \times V \to \mathbb{F}$. Let $B = \{b_1, \ldots, b_n\}$ be a basis of V and let

A be the Gram matrix of $\langle , \rangle$ with respect to the basis B.

Let $u = u_1 b_1 + \cdots + u_n b_n \in V$ and let $v = v_1 b_1 + \cdots + v_n b_n \in V$ so that

$$[u]_B = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \quad \text{and} \quad [v]_B = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}.$$

Then

$$\begin{aligned} \langle u, v \rangle &= \langle u_1 b_1 + \cdots + u_n b_n, v_1 b_1 + \cdots + v_n b_n \rangle \\ &= \sum_{i,j=1}^{n} u_i \overline{v_j} \langle b_i, b_j \rangle = \sum_{i,j=1}^{n} u_i A_{ij} \overline{v_j} \\ &= [u]_B^t A [\bar{v}]_B. \end{aligned}$$

Example IP4. The $\mathbb{R}$-vector space $\mathbb{R}[x]_{\leqslant 2}$ has basis $B = \{1, x, x^2\}$. Since the standard inner product $\langle\ |\ \rangle\colon \mathbb{R}[x]_{\leqslant 2} \times \mathbb{R}[x]_{\leqslant 2} \to \mathbb{R}$ has

$$\langle p|q\rangle = \int_0^1 pq\,dx$$

then

$$\begin{array}{ccc} \langle 1|1\rangle = 1, & \langle 1|x\rangle = \frac{1}{2}, & \langle 1|x^2\rangle = \frac{1}{3}, \\ \langle x|1\rangle = \frac{1}{2}, & \langle x|x\rangle = \frac{1}{3}, & \langle x|x^2\rangle = \frac{1}{4}, \\ \langle x^2|1\rangle = \frac{1}{3}, & \langle x^2|x\rangle = \frac{1}{4}, & \langle x^2|x^2\rangle = \frac{1}{5}, \end{array}$$

then the Gram matrix of $\langle\ |\ \rangle$ with respect fo B is

$$A = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}$$

If $u = 7 + 3x + 2x^2$ and $v = 5 + x^2$ then

$$[u]_B = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \quad \text{and} \quad [v]_B = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$\begin{aligned}
\langle u|v\rangle &= \int_0^1 (7+3x+2x^2)(5+x^2)\,dx \\
&= \int_0^1 (35+7x^2+15x+3x^3+10x^2+2x^4)\,dx \\
&= 35+\tfrac{7}{3}+\tfrac{15}{2}+\tfrac{3}{4}+\tfrac{10}{3}+\tfrac{2}{5}
\end{aligned}$$

and

$$\begin{aligned}
[u]_B^t A[v]_B &= \begin{pmatrix} 7 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 7 & 3 & 2 \end{pmatrix} \begin{pmatrix} 5+\frac{1}{3} \\ \frac{5}{2}+\frac{1}{4} \\ \frac{5}{3}+\frac{1}{5} \end{pmatrix} \\
&= 35+\tfrac{7}{3}+\tfrac{15}{2}+\tfrac{3}{4}+\tfrac{10}{3}+\tfrac{2}{5}.
\end{aligned}$$

So $\langle u|v\rangle = [u]_B^t A[v]_B$.